Osnovne enačbe nelinearne mehanike deformabilnih teles

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1. Deformacije (kinematične enačbe)

1.1. Materialni in prostorski opis gibanja in deformiranja telesa

1.1.1. Materialne ali telesne koordinate

$$x^0 \equiv x_1^0, \qquad y^0 \equiv x_2^0, \qquad z^0 \equiv x_3^0. \tag{1}$$

1.1.2. Nedeformirani bazni vektorji

$\vec{e}_1 \equiv \vec{e}_x$, $\vec{e}_2 \equiv \vec{e}_y$ in $\vec{e}_3 \equiv \vec{e}_z$.

1.1.3. Pomiki

Pomike v smereh $\vec{e}_1$, $\vec{e}_2$ in $\vec{e}_3$ označimo z u, v in w. Lahko jih izrazimo v materialnem ali v prostorskem opisu.

1.1.4. Prostorske koordinate

$$\begin{aligned} x \equiv x_1 &= x_1^0 + u, \\ y \equiv x_2 &= x_2^0 + v, \\ z \equiv x_3 &= x_3^0 + w. \end{aligned} \tag{2}$$

1.1.5. Materialni opis

$$\begin{aligned} x_1\left(x_1^0, x_2^0, x_3^0\right) &= x_1^0 + u\left(x_1^0, x_2^0, x_3^0\right), \\ x_2\left(x_1^0, x_2^0, x_3^0\right) &= x_2^0 + v\left(x_1^0, x_2^0, x_3^0\right), \\ x_3\left(x_1^0, x_2^0, x_3^0\right) &= x_3^0 + w\left(x_1^0, x_2^0, x_3^0\right). \end{aligned} \tag{3}$$

1.1.6. Prostorski opis

$$\begin{aligned} x_1^0\left(x_1, x_2, x_3\right) &= x_1 - \tilde{u}\left(x_1, x_2, x_3\right), \\ x_2^0\left(x_1, x_2, x_3\right) &= x_2 - \tilde{v}\left(x_1, x_2, x_3\right), \\ x_3^0\left(x_1, x_2, x_3\right) &= x_3 - \tilde{w}\left(x_1, x_2, x_3\right). \end{aligned} \tag{4}$$

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1.1.7. Deformacijski gradient F (materialni opis)

$$\begin{aligned} F &= \frac{\partial \vec{r}}{\partial \vec{r^0}} \\ &= \begin{bmatrix} \frac{\partial x_1}{\partial x_1^0} & \frac{\partial x_1}{\partial x_2^0} & \frac{\partial x_1}{\partial x_3^0} \\ \frac{\partial x_2}{\partial x_1^0} & \frac{\partial x_2}{\partial x_2^0} & \frac{\partial x_2}{\partial x_3^0} \\ \frac{\partial x_3}{\partial x_1^0} & \frac{\partial x_3}{\partial x_2^0} & \frac{\partial x_3}{\partial x_3^0} \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{bmatrix} \\ &= \begin{bmatrix} 1 + \frac{\partial u}{\partial x_0} & \frac{\partial u}{\partial y_0} & \frac{\partial u}{\partial z_0} \\ \frac{\partial v}{\partial x_0} & 1 + \frac{\partial v}{\partial y_0} & \frac{\partial v}{\partial z_0} \\ \frac{\partial w}{\partial x_0} & \frac{\partial w}{\partial y_0} & 1 + \frac{\partial w}{\partial z_0} \end{bmatrix} = I + \nabla^0 \vec{u}. \end{aligned} \tag{5}$$

Opomba: Nekateri deformacijski gradient definirajo s transponirano preslikavo F^T.

1.1.8. Inverz deformacijskega gradienta F^{-1} (prostorski opis)

$$
\begin{aligned}
F^{-1} &= \frac{\partial \vec{r}^0}{\partial \vec{r}} \\
&= \begin{bmatrix} \frac{\partial x_1^0}{\partial x_1} & \frac{\partial x_1^0}{\partial x_2} & \frac{\partial x_1^0}{\partial x_3} \\ \frac{\partial x_2^0}{\partial x_1} & \frac{\partial x_2^0}{\partial x_2} & \frac{\partial x_2^0}{\partial x_3} \\ \frac{\partial x_3^0}{\partial x_1} & \frac{\partial x_3^0}{\partial x_2} & \frac{\partial x_3^0}{\partial x_3} \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial x_0}{\partial x} & \frac{\partial x_0}{\partial y} & \frac{\partial x_0}{\partial z} \\ \frac{\partial y_0}{\partial x} & \frac{\partial y_0}{\partial y} & \frac{\partial y_0}{\partial z} \\ \frac{\partial z_0}{\partial x} & \frac{\partial z_0}{\partial y} & \frac{\partial z_0}{\partial z} \end{bmatrix} \\
&= \begin{bmatrix} 1 - \frac{\partial \tilde{u}}{\partial x} & \frac{\partial \tilde{u}}{\partial y} & \frac{\partial \tilde{u}}{\partial z} \\ \frac{\partial \tilde{v}}{\partial x} & 1 - \frac{\partial \tilde{v}}{\partial y} & \frac{\partial \tilde{v}}{\partial z} \\ \frac{\partial \tilde{w}}{\partial x} & \frac{\partial \tilde{w}}{\partial y} & 1 - \frac{\partial \tilde{w}}{\partial z} \end{bmatrix} = I - \nabla \vec{u}.
\end{aligned} \tag{6}
$$

1.1.9. Deformirani bazni vektorji $\vec{g}_i$ in $\vec{G}_i$

Izhajamo iz enačb

$$
\begin{aligned}
\vec{dr} &= \sum_i dx_i\, \vec{e}_i = \sum_i dx_i^0\, \vec{g}_i, \\
\vec{dr}^0 &= \sum_i dx_i^0\, \vec{e}_i = \sum_i dx_i\, \vec{G}_i,
\end{aligned} \tag{7}
$$

upoštevamo zvezo

$$\vec{dr} = F\, \vec{dr}^0 \tag{8}$$

in dobimo

$$\vec{g}_i = F\, \vec{e}_i, \tag{9a}$$

$$\vec{G}_i = F^{-1}\, \vec{e}_i. \tag{9b}$$

1.2. Cauchy Greenova tenzorja

1.2.1. Levi Cauchy Greenov tenzor C (materialni opis)

$$C = F^T\, F. \tag{10}$$

1.2.2. Desni Cauchy Greenov tenzor B

$$B = F\, F^T. \tag{11}$$

V prostorskem opisu najprej izračunamo inverz B^{-1} po enačbi

$$B^{-1} = F^{-T}\, F^{-1}. \tag{12}$$

1.3. Polarna razcepa deformacijskega gradienta

1.3.1. RU razcep deformacijskega gradienta F

$$F = R\, U, \tag{13a}$$

$$U = \sqrt{C} = \sqrt{F^T\, F} = \text{sqrtm(C) MATLAB}, \tag{13b}$$

$$R = F\, U^{-1}. \tag{13c}$$

1.3.2. VR razcep deformacijskega gradienta F

$$F = V\, R, \tag{14a}$$

$$V = \sqrt{B} = \sqrt{F\, F^T} = \text{sqrtm(B) MATLAB}, \tag{14b}$$

$$R = V^{-1}\, F. \tag{14c}$$

Zveze med preslikavama U in V:

$$U = R^T\, V\, R, \tag{15a}$$

$$V = R\, U\, R^T. \tag{15b}$$

1.4. Spremembe dolžine, površine in volumna

1.4.1. Sprememba dolžine

Sprememba dolžin vektorjev $\vec{da}^0$ in $\vec{da}$:

$$ds^2 = \vec{da} \cdot \vec{da} = \vec{da}^0 \cdot C\, \vec{da}^0, \tag{16a}$$

$$(ds^0)^2 = \vec{da}^0 \cdot \vec{da}^0 = \vec{da} \cdot B^{-1}\, \vec{da}. \tag{16b}$$

1.4.2. Sprememba površine (Nansonova formula)

Sprememba površine dS^0 ploskve z normalo $\vec{n}^0$

$$dS\, \vec{n} = \vec{dS} = J\, F^{-T}\, \vec{dS}^0 = dS^0\, J\, F^{-T}\, \vec{n}^0. \tag{17a}$$

1.4.3. Sprememba volumna

Sprememba volumna dV^0:

$$dV = J\, dV^0. \tag{18a}$$

1.5. Tenzorji deformacij

1.5.1. Green Lagrangev tenzor E

Green Lagrangev tenzor E (velikih deformacij v materialnem opisu)

$$E = \frac{1}{2}(C - I) \tag{19a}$$

$$= \frac{1}{2}\left(F^T\, F - I\right) \tag{19b}$$

$$= \frac{1}{2}\left(\nabla^0\vec{u} + \left(\nabla^0\vec{u}\right)^T + \left(\nabla^0\vec{u}\right)^T\, \left(\nabla^0\vec{u}\right)\right). \tag{19c}$$

$$E_{ij} = \frac{1}{2}\left(\vec{g}_i \cdot \vec{g}_j - \vec{e}_i \cdot \vec{e}_j\right) \tag{19d}$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j^0} + \frac{\partial u_j}{\partial x_i^0} + \sum_k \frac{\partial u_k}{\partial x_i^0}\frac{\partial u_k}{\partial x_j^0}\right). \tag{19e}$$

1.5.2. Euler Almansijev tenzor e

Euler Almansijev tenzor e (velikih deformacij v prostorskem opisu)

$$e = \frac{1}{2}(I - B^{-1}) \tag{20a}$$

$$= \frac{1}{2}\left(I - F^{-T}\, F^{-1}\right) \tag{20b}$$

$$= \frac{1}{2}\left(\nabla\vec{u} + (\nabla\vec{u})^T - (\nabla\vec{u})^T\, (\nabla\vec{u})\right). \tag{20c}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \sum_k \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right). \tag{20d}$$

Zveze med Green Lagrangevim tenzorjem E in Euler Almansijevim tenzorjem e:

$$E = F^T\, e\, F, \tag{21a}$$
$$e = F^{-T}\, E\, F^{-1}. \tag{21b}$$

1.5.3. Tenzor majhnih deformacij ε

Green Lagrangev tenzor ε (majhnih deformacij v materialnem opisu)

$$\varepsilon = \frac{1}{2}(F + F^T - 2\, I) \tag{22a}$$
$$= \frac{1}{2}\left(\nabla^0 \vec{u} + \left(\nabla^0 \vec{u}\right)^T\right). \tag{22b}$$
$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j^0} + \frac{\partial u_j}{\partial x_i^0}\right). \tag{22c}$$

1.5.4. Tenzor rotacij (psevdorotacij) ω

1.6. Pomembne deformacijske količine

1.6.1. Specifična sprememba dolžine

Specifična sprememba dolžine v smeri enotskega vektorja $\vec{n^0} \equiv \vec{e_a}$ (materialni opis):

$$D_{aa} = \frac{ds - ds^0}{ds^0} = \sqrt{1 + 2\, E_{aa}} - 1, \tag{23a}$$
$$= \sqrt{\vec{n^0} \cdot C\, \vec{n^0}} - 1. \tag{23b}$$

Velja zveza [1],[2]

$$E_{aa} = D_{aa} + \frac{D_{aa}^2}{2}. \tag{24}$$

1.6.2. Sprememba pravega kota

Sprememba pravega kota med vektorjema $\vec{a^0} = a^0 \vec{n^0}$ in $\vec{b^0} = b^0 \vec{m^0}$ (materialni opis):

$$D_{ab} \equiv \Delta\theta = \theta^0 - \theta = \frac{\pi}{2} - \theta, \tag{25a}$$
$$\sin(D_{ab}) = \frac{2\, E_{ab}}{\sqrt{1 + 2\, E_{aa}}\, \sqrt{1 + 2\, E_{bb}}}, \tag{25b}$$
$$\sin(D_{ab}) = \frac{\vec{n^0} \cdot C\, \vec{m^0}}{\sqrt{\vec{n^0} \cdot C\, \vec{n^0}}\, \sqrt{\vec{m^0} \cdot C\, \vec{m^0}}}. \tag{25c}$$

1.6.3. Stretch (razteg ali skrčitev)

Stretch (razteg ali skrčitev) λ v smeri enotskega vektorja $\vec{m^0}$:

$$\lambda = \frac{ds}{ds^0} = \sqrt{\vec{m^0} \cdot C\, \vec{m^0}} = \frac{1}{\sqrt{\vec{m} \cdot B^{-1}\, \vec{m}}}, \tag{26}$$

kjer sta enotska vektorja $\vec{m}$ in $\vec{m^0}$ povezana z enačbo

$$\lambda\, \vec{m} = F\, \vec{m^0}. \tag{27}$$

1.6.4. Glavni raztegi ali skrčitve

Glavne raztege ali skrčitve λ_i in pripadajoče glavne smeri (v materialnem opisu) $\vec{m_i^0}$ dobimo kot lastne vrednosti in lastne vektorje iz enačb:

$$\left(C - \lambda_i^2\, I\right) \vec{m_i^0} = \vec{0}, \tag{28a}$$
$$\left(F^T\, F - \lambda_i^2\, I\right) \vec{m_i^0} = \vec{0}, \tag{28b}$$
$$\left(U^T\, U - \lambda_i^2\, I\right) \vec{m_i^0} = \vec{0}, \tag{28c}$$
$$\left(U^2 - \lambda_i^2\, I\right) \vec{m_i^0} = (U + \lambda_i\, I)\, (U - \lambda_i\, I)\, \vec{m_i^0} = \vec{0}, \tag{28d}$$
$$(U - \lambda_i\, I)\, \vec{m_i^0} = \vec{0}. \tag{28e}$$

Predelava enačb za razteg v prostorskem opisu: Iz enačbe (27) dobimo

$$\vec{m^0} = \lambda\, F^{-1} \vec{m}. \tag{29}$$

V enačbi (28b) upoštevamo zvezo (29) in nato pa pomnožimo enačbo (28b) še z leve z F in dobimo

$$\left(F\, F^T - \lambda_i^2\, I\right) \vec{m_i} = \vec{0}.$$

Glavne raztege ali skrčitve λ_i in pripadajoče glavne smeri (v prostorskem opisu) $\vec{m_i}$ dobimo kot lastne vrednosti in lastne vektorje iz enačb:

$$\left(B - \lambda_i^2\, I\right) \vec{m_i} = \vec{0}, \tag{30a}$$
$$\left(F\, F^T - \lambda_i^2\, I\right) \vec{m_i} = \vec{0}, \tag{30b}$$
$$\left(V\, V^T - \lambda_i^2\, I\right) \vec{m_i} = \vec{0}, \tag{30c}$$
$$\left(V^2 - \lambda_i^2\, I\right) \vec{m_i} = (V + \lambda_i\, I)\, (V - \lambda_i\, I)\, \vec{m_i} = \vec{0}, \tag{30d}$$
$$(V - \lambda_i\, I)\, \vec{m_i} = \vec{0}. \tag{30e}$$

Zveza med pripadajočimi glavnimi smermi (v materialnem opisu) $\vec{m_i^0}$ in in pripadajočimi glavnimi smermi (v prostorskem opisu) $\vec{m_i}$:

$$\vec{m_i} = R\, \vec{m_i^0}, \tag{31a}$$
$$\vec{m_i^0} = R^T\, \vec{m_i}. \tag{31b}$$

Enačbi (28e) in (30e) sledita neposredno iz enačb (28d) in (30d) ob upoštevanju pozitivne definitnosti (nesingularnosti) preslikav U in V.

1.7. Osnovni pojmi tenzorske algebre

1.7.1. Operacije nad tenzorji

Nekaj operacij nad tenzorji: (A in B sta tenzorja 2 reda)

(.) $(\lambda\, A)\, \vec{u} = \lambda\, (A\, \vec{u})$, $(A\, B)\, \vec{u} = A\, (B\, \vec{u})$,

(+) $(A + B)\, \vec{u} = A\, \vec{u} + B\, \vec{u}$,

(T) $\vec{v} \cdot (A^T\, \vec{u}) = \vec{u} \cdot (A\, \vec{v})$,

(tr) $\mathrm{tr}(A) = I_1^A$, I_1^A je prva invarianta tenzorja A (vsota diagonalnih členov v matriki $[A_{ij}]$).

(:) $A : B = \mathrm{tr}(A^T\, B)$.

1.7.2. Tenzorski produkt vektorjev

S predpisom

$$(\vec{u} \otimes \vec{v})\,\vec{w} = (\vec{w} \cdot \vec{v})\,\vec{u}. \tag{32}$$

definiramo linearno preslikavo z imenom *tenzorski produkt vektorjev* $\vec{u} \otimes \vec{v}$.

Nekaj lastnosti tenzorskega produkta:

$$\begin{aligned}
(\vec{u} \otimes \vec{v})\,(\alpha\vec{w} + \vec{x}) &= \alpha\,(\vec{u} \otimes \vec{v})\,\vec{w} + (\vec{u} \otimes \vec{v})\,\vec{x}, && (33a)\\
(\alpha\,\vec{u} + \beta\,\vec{v}) \otimes \vec{w} &= \alpha\,(\vec{u} \otimes \vec{w}) + \beta\,(\vec{v} \otimes \vec{w}), && (33b)\\
(\vec{u} \otimes \vec{v})\,(\vec{w} \otimes \vec{x}) &= (\vec{v} \cdot \vec{w})\,(\vec{u} \otimes \vec{x}), && (33c)\\
\mathrm{tr}(\vec{u} \otimes \vec{v}) &= \vec{u} \cdot \vec{v}, && (33d)\\
A_{ij} = \vec{e}_i \cdot A\,\vec{e}_j \longrightarrow A &= \sum_{i,j} A_{ij}\,(\vec{e}_i \otimes \vec{e}_j), && (33e)\\
A : B &= \sum_{i,j} A_{ij}\,B_{ij}, && (33f)\\
A : (\vec{u} \otimes \vec{v}) &= \vec{u} \cdot A\,\vec{v} = (\vec{u} \otimes \vec{v}) : A, && (33g)\\
(\vec{u} \otimes \vec{v}) : (\vec{w} \otimes \vec{x}) &= (\vec{u} \cdot \vec{w})\,(\vec{v} \cdot \vec{x}). && (33h)
\end{aligned}$$

Naj bo

$$\begin{aligned}
A &= (\vec{u} \otimes \vec{v}),\\
\vec{u} &= \sum_i u_i\,\vec{e}_i,\\
\vec{v} &= \sum_i v_i\,\vec{e}_i.
\end{aligned}$$

Potem velja

$$\begin{aligned}
A_{ij} &= \vec{u}_i \cdot A\,\vec{e}_j\\
&= u_i\,v_j.
\end{aligned}$$

1.8. Spektralni razcep simetričnega tenzorja

1.8.1. Izrek o spektralnem razcepu

Izrek o spektralnem razcepu: Naj bo S simetrični tenzor. Tedaj obstajajo trije ortonormirani lastni vektorji $\vec{m}_i$ in tri realne lastne vrednosti λ_i (ne nujno različne med seboj). Potem imamo tri možnosti:

- $\lambda_1 < \lambda_2 < \lambda_3$: $\quad S = \sum_{i=1}^{3} \lambda_i\,(\vec{m}_i \otimes \vec{m}_i)$.
- $\lambda_1 \neq \lambda_2 = \lambda_3$: $\quad S = \lambda_1\,(\vec{m}_1 \otimes \vec{m}_1) + \lambda_2\,(I - (\vec{m}_1 \otimes \vec{m}_1))$.
- $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$: $\quad S = \lambda\,I$.

1.8.2. Spektralni razcepi tenzorjev deformacij

Spektralni razcepi tenzorjev U, C, V, B, E in e:

$$\begin{aligned}
U &= \sum_{i=1}^{3} \lambda_i\,(\vec{m}_i^0 \otimes \vec{m}_i^0), && (34a)\\
C &= \sum_{i=1}^{3} \lambda_i^2\,(\vec{m}_i^0 \otimes \vec{m}_i^0), && (34b)\\
V &= \sum_{i=1}^{3} \lambda_i\,(\vec{m}_i \otimes \vec{m}_i), && (34c)\\
B &= \sum_{i=1}^{3} \lambda_i^2\,(\vec{m}_i \otimes \vec{m}_i), && (34d)\\
E &= \frac{1}{2} \sum_{i=1}^{3} (\lambda_i^2 - 1)\,(\vec{m}_i^0 \otimes \vec{m}_i^0), && (34e)\\
e &= \frac{1}{2} \sum_{i=1}^{3} \frac{(\lambda_i^2 - 1)}{\lambda_i^2}\,(\vec{m}_i \otimes \vec{m}_i). && (34f)
\end{aligned}$$

1.8.3. Razcep deformacijskega gradienta

Spektralni razcep ne obstaja. Obstaja pa spodnji razcep deformacijskega gradienta F:

$$F = \sum_{i=1}^{3} \lambda_i\,(\vec{m}_i \otimes \vec{m}_i^0), \tag{35}$$

1.8.4. Posplošeni tenzorji deformacij

Posplošeni tenzorji deformacij:

$$\begin{aligned}
E^{(n)} &= \frac{1}{n} \sum_{i=1}^{3} (\lambda_i^n - 1)\,(\vec{m}_i^0 \otimes \vec{m}_i^0), && (36a)\\
e^{(n)} &= \frac{1}{n} \sum_{i=1}^{3} (1 - \lambda_i^{-n})\,(\vec{m}_i \otimes \vec{m}_i). && (36b)
\end{aligned}$$

1.9. Materialni odvodi po času

1.9.1. Materialni odvod funkcije f po času

$$\frac{Df}{Dt} = \left.\frac{\partial f(\vec{r}^0, t)}{\partial t}\right|_{\vec{r}^0 = const} = \dot{f} \tag{37}$$

1.9.2. Lokalni relativni odvod po času

$$\frac{\partial f}{\partial t} = \left.\frac{\partial f(\vec{r}, t)}{\partial t}\right|_{\vec{r} = const} \tag{38}$$

1.9.3. Hitrost delca telesa

$$\begin{aligned}
\vec{v}(\vec{r}^0, t) &= \frac{D\vec{r}}{Dt} = \left.\frac{\partial \vec{r}(\vec{r}^0, t)}{\partial t}\right|_{\vec{r}^0 = const} = \dot{\vec{r}}\\
&= \frac{D(\vec{r}^0 + \vec{u})}{Dt} = \left.\frac{\partial (\vec{r}^0 + \vec{u})(\vec{r}^0, t)}{\partial t}\right|_{\vec{r}^0 = const}\\
&= \left.\frac{\partial \vec{u}(\vec{r}^0, t)}{\partial t}\right|_{\vec{r}^0 = const} = \dot{\vec{u}}
\end{aligned} \tag{39}$$

1.9.4. Pospešek delca telesa

$$\vec{a}(\vec{r^0},t) = \frac{D\vec{v}}{Dt} = \left.\frac{\partial \vec{v}(\vec{r^0},t)}{\partial t}\right|_{\vec{r^0}=const} = \dot{\vec{v}} = \ddot{\vec{r}} \tag{40}$$

1.10. Materialni odvod funkcije f v prostorskih koordinatah

$$f = f(x_i,t) = f(x_i(x_k^0,t),t). \tag{41}$$

$$\begin{aligned}\frac{Df}{Dt} &= \left.\frac{\partial f(x_i(x_k^0,t),t)}{\partial t}\right|_{x_k^0=const} \\ &= \left.\frac{\partial f(x_i,t)}{\partial t}\right|_{x_i=const} + \sum_i \left.\frac{\partial f(x_i,t)}{\partial x_i}\right|_{t=const} \left.\frac{\partial x_i(x_k^0,t)}{\partial t}\right|_{x_i^0=const} \\ &= \left.\frac{\partial f(x_i,t)}{\partial t}\right|_{x_i=const} + \sum_i \left.\frac{\partial f(x_i,t)}{\partial x_i}\right|_{t=const} v_i \\ &= \frac{\partial f}{\partial t} + \sum_i v_i \frac{\partial f}{\partial x_i}.\end{aligned} \tag{42}$$

1.10.1. Materialni odvod v operatorski obliki

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}. \tag{43}$$

1.10.2. Hitrost delca telesa v prostorski obliki

$$\vec{v} = \frac{D\vec{r}}{Dt} = \sum_i v_i\,\vec{e}_i. \tag{44}$$

1.10.3. Pospešek delca telesa v prostorski obliki

$$\begin{aligned}\vec{a} &= \frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\,\vec{v}, \\ \frac{Dv_i}{Dt} &= \frac{\partial v_i}{\partial t} + \sum_k v_k\frac{\partial v_i}{\partial x_k}.\end{aligned} \tag{45}$$

1.10.4. Hitrostni gradient L

$$L = \frac{\partial\vec{v}}{\partial\vec{r}} = [L_{ij}] = \left[\frac{\partial v_i}{\partial x_j}\right]. \tag{46}$$

1.10.5. Materialni odvod deformacijskega gradienta po času

$$\frac{DF}{Dt} = \left.\frac{\partial F(\vec{r^0},t)}{\partial t}\right|_{\vec{r^0}=const} = \dot{F} = L\,F. \tag{47}$$

Idejna izpeljava:

$$\begin{aligned}\frac{DF}{Dt} &= \left.\frac{\partial F}{\partial t}\right|_{\vec{r^0}} = \left.\frac{\partial}{\partial t}\right|_{\vec{r^0}}\frac{\partial\vec{r}}{\partial\vec{r^0}} = \frac{\partial}{\partial\vec{r^0}}\left.\frac{\partial\vec{r}}{\partial t}\right|_{\vec{r^0}} \\ &= \frac{\partial\vec{v}}{\partial\vec{r^0}} = \frac{\partial\vec{v}}{\partial\vec{r}}\frac{\partial\vec{r}}{\partial\vec{r^0}} = \frac{\partial\vec{v}}{\partial\vec{r}}F \\ &= L\,F.\end{aligned} \tag{48}$$

1.11. Hitrost deformacijskega gradienta D in spin W

$$L = D + W. \tag{49}$$

1.11.1. Hitrost deformacijskega gradienta D

$$D = \frac{1}{2}(L + L^T). \tag{50}$$

1.11.2. Spin W

$$W = \frac{1}{2}(L - L^T). \tag{51}$$

1.12. Hitrosti deformacijskih tenzorjev

1.12.1. Hitrost desnega Cauchy Greenovega tenzorja $\dot{C}$

$$\dot{C} = 2\,F^T\,D\,F. \tag{52}$$

1.12.2. Hitrost levega Cauchy Greenovega tenzorja $\dot{B}$

$$\dot{B} = L\,B + B\,L^T. \tag{53}$$

1.12.3. Hitrost Green Lagrangevega tenzorja $\dot{E}$

$$\dot{E} = F^T\,D\,F. \tag{54}$$

1.12.4. Hitrost Euler Almansijevega tenzorja $\dot{e}$

$$\dot{e} = \frac{1}{2}(B^{-1}\,L + L^T\,B^{-1}) = \frac{1}{2}\,B^{-1}\,\dot{B}\,B^{-1}. \tag{55}$$

1.13. Fizikalni pomen D in W

1.13.1. Fizikalni pomen D

$$\dot{\lambda} = \lambda\,\vec{n}\cdot D\,\vec{n} \to (\dot{\ln\lambda}) = \vec{n}\cdot D\,\vec{n} \tag{56}$$

1.13.2. Fizikalni pomen W

$$\begin{aligned}\dot{\lambda}\,\vec{n} + \lambda\,\dot{\vec{n}} = \lambda\,L\,\vec{n} \to \dot{\vec{n}} = \left(L - \frac{\dot{\lambda}}{\lambda}I\right)\vec{n} = \left(D - \frac{\dot{\lambda}}{\lambda}I\right)\vec{n} + W\,\vec{n}, \\ \vec{m}_D \in \mathrm{eig}(D) \to \dot{\vec{m}}_D = W\,\vec{m}_D = \vec{\omega}\times\vec{m}_D.\end{aligned} \tag{57}$$

1.14. Materialni odvodi nekaterih geometrijskih količin

1.14.1. Materialni odvod kvadrata elementarne dolžine

$$\frac{D(ds^2)}{Dt} = 2\,d\vec{r}\cdot D\,d\vec{r} = 2\,d\vec{r^0}\cdot\dot{E}\,d\vec{r^0}. \tag{58}$$

1.14.2. Materialni odvod elementarne ploskve

$$\frac{D(\vec{n}\,dS)}{Dt} = (\mathrm{div}\,\vec{v} - L^T)\,\vec{n}\,dS. \tag{59}$$

1.14.3. Materialni odvod elementarne prostornine

$$\frac{D(dV)}{Dt} = (\mathrm{div}\,\vec{v})\,dV = \sum_i\frac{\partial v_i}{\partial x_i}\,dV \tag{60}$$

2. Napetosti (ravnotežne enačbe)

2.1. Osnovni pojmi

2.1.1. Rezultanta površinske obtežbe

$$d\vec{F} = \vec{t}\,dS \rightarrow \vec{F} = \int_S \vec{t}\,dS \tag{61}$$

2.1.2. Napetostni vektor $\vec{t}$ in Cauchyjev napetostni tenzor σ

$$\begin{aligned}\vec{t} &= \vec{t}(\vec{r}, t, \vec{n})\\ &= \sum_{i,j} \sigma_{ij}\,(\vec{e}_j \otimes \vec{e}_i)\,\vec{n}\\ &= \sum_{i,j} \sigma_{ij}\,(\vec{e}_i \cdot \vec{n})\,\vec{e}_j\\ \sigma &= \sum_{i,j} \sigma_{ij}\,\vec{e}_i \otimes \vec{e}_j\end{aligned} \tag{62}$$

2.1.3. Zasuk R - objektivnost tenzorja napetosti (je objektiven)

$$\begin{gathered}\vec{n}^+ = R\,\vec{n} \rightarrow \vec{n} = R^T\,\vec{n}^+,\\ \vec{t}^+ = R\,\vec{t} \rightarrow \vec{t} = R^T\,\vec{t}^+,\\ \vec{t} = R^T\,\vec{t}^+ = \sigma^T\,\vec{n} = \sigma\,\vec{n} = \sigma\,R^T\,\vec{n}^+ \rightarrow \vec{t}^+ = R\,\sigma\,R^T\,\vec{n}^+ = \sigma^+\vec{n}^+.\end{gathered} \tag{63}$$

2.1.4. Prvi Piola Kirchhoffov tenzor napetosti P (PK1)

$$d\vec{F} = \vec{t}\,dS = \sigma\,\vec{n}\,dS = \sigma\,J\,F^{-T}\,\vec{n}^0\,dS_0 = P\,\vec{n}^0\,dS_0. \tag{64}$$

Pripadajoči napetostni vektor označimo s $\vec{p}^0 = P\,\vec{n}^0$. Zveza med $\vec{t}$ in $\vec{p}^0$

$$\vec{t} = \frac{dS_0}{dS}\,\vec{p}^0 \tag{65}$$

$$\begin{aligned}\vec{p}^0_1 &= \vec{p}^0(\vec{e}_1) = P\,\vec{e}_1\\ \vec{p}^0_2 &= \vec{p}^0(\vec{e}_2) = P\,\vec{e}_2\\ \vec{p}^0_3 &= \vec{p}^0(\vec{e}_3) = P\,\vec{e}_3\\ \vec{p}^0_i &= \sum_{i=1}^{3} P_{ji}\,\vec{e}_j\end{aligned} \tag{66}$$

2.1.5. Drugi Piola Kirchhoffov tenzor napetosti S (PK2)

$$\begin{aligned}S &= F^{-1}\,P\\ &= F^{-1}\,J\,\sigma\,F^{-T}\\ &= S^T\end{aligned} \tag{67}$$

Pripadajoči napetostni vektor označimo s $\vec{s}^0 = S\,\vec{n}^0$. σ dobimo z obratom

$$\sigma = \frac{1}{J}\,F\,S\,F^T. \tag{68}$$

2.1.6. Kirchhoffov tenzor napetosti τ

$$\begin{aligned}\tau &= J\,\sigma\\ &= F\,S\,F^T\\ &= \tau^T\\ &= P\,F^T.\end{aligned} \tag{69}$$

2.1.7. Biotov tenzor napetosti T_B

$$\begin{gathered}T_B = R^T\,P\\ = R^T\,F\,S\\ R\,T_B = P\\ d\vec{F} = R\,T_B\,\vec{n}^0\,dS_0 \rightarrow R^T\,d\vec{F} = T_B\,\vec{n}^0\,dS_0 = d\,\vec{F}_B\end{gathered} \tag{70}$$

2.1.8. Tabela tenzorjev napetosti

	σ	P	S	simetrija
σ		$\frac{1}{J}\,P\,F^T$	$\frac{1}{J}\,F\,S\,F^T$	$\sigma = \sigma^T$
P	$J\,\sigma\,F^{-T}$		$F\,S$	$P \neq P^T$
S	$J\,F^{-1}\,\sigma\,F^{-T}$	$F^{-1}\,P$		$S = S^T$
τ	$J\,\sigma$	$P\,F^T$	$F\,S\,F^T$	$\tau = \tau^T$
T_B	$J\,R^T\,\sigma\,F^{-T}$	$R^T\,P$	$R^T\,F\,S$	$T_B \neq T_B^T$

2.1.9. Objektivnost tenzorjev napetosti (vsi so objektivni)

$$\begin{aligned}\sigma^+ &= Q\,\sigma\,Q^T,\ \text{objektiven prostorski},\\ P^+ &= Q\,P\ \text{objektiven mesani},\\ S^+ &= S\ \text{objektiven materialni},\\ \tau^+ &= Q\,\tau\,Q^T,\ \text{objektiven prostorski},\\ T_B^+ &= T_B\ \text{objektiven materialni},\end{aligned} \tag{71}$$

2.2. Ravnotežni pogoji

2.2.1. Cauchy

$$\operatorname{div}\sigma + \rho\vec{b} = \rho\,\frac{D\,\vec{v}}{D\,t} \tag{72a}$$

ali

$$\frac{\partial\sigma_{ij}}{\partial x_j} + \rho\,b_i = \rho\,\frac{D\,v_i}{D\,t} \tag{72b}$$

Robni pogoji:

$$\vec{t} = \sigma\,\vec{n} \tag{72c}$$

2.2.2. PK1

$$\operatorname{div}_0 P + \rho_0\vec{b} = \rho_0\,\frac{D\,\vec{v}}{D\,t} \tag{73a}$$

ali

$$\frac{\partial P_{ij}}{\partial x^0_j} + \rho_0\,b_i = \rho_0\,\frac{D\,v_i}{D\,t} \tag{73b}$$

Momentni pogoj:

$$P\,F^T - F\,P^T = 0 \tag{73c}$$

Od devetih enačb so samo tri neodvisne.

Robni pogoji:

$$\vec{p}^0 = P\,\vec{n}^0. \tag{73d}$$

2.2.3. PK2

$$\mathrm{div}_0\left((I + \nabla\,\vec{u})\,S\right) + \rho_0\vec{b} = \rho_0\,\frac{D\,\vec{v}}{D\,t} \tag{74a}$$

ali

$$\frac{\partial}{\partial x^0_k}\left(\left(\delta_{ij} + \frac{\partial u_i}{\partial x^0_j}\right) S_{kj}\right) + \rho_0\,b_i = \rho_0\,\frac{D\,v_i}{D\,t} \tag{74b}$$

ali

$$\frac{\partial}{\partial x^0_k}\left(F_{ij}\,S_{kj}\right) + \rho_0\,b_i = \rho_0\,\frac{D\,v_i}{D\,t} \tag{74c}$$

Robni pogoji:

$$\vec{p}^0 = F\,S\,\vec{n}^0. \tag{74d}$$

3. Princip virtualnega dela (PVD) in princip virtualne moči (PVM)

3.1. Prostorski opis

$$\vec{f} = \rho\,(\vec{b} - \dot{\vec{v}}). \tag{75a}$$

$$\int_V \sigma : \delta D\,dV = \int_S \vec{t}\cdot\delta\vec{v}\,dS + \int_V \vec{f}\cdot\delta\vec{v}\,dV. \tag{75b}$$

$$\int_V \sigma : \delta\varepsilon\,dV = \int_S \vec{t}\cdot\delta\vec{u}\,dS + \int_V \vec{f}\cdot\delta\vec{u}\,dV. \tag{75c}$$

$$\delta\varepsilon = \frac{1}{2}\left(\frac{\partial\delta\vec{u}}{\partial\vec{r}} + \left(\frac{\partial\delta\vec{u}}{\partial\vec{r}}\right)^T\right) = \delta\varepsilon^T. \tag{75d}$$

3.2. Materialni opis

$$\int_{V^0} P : \delta\dot{F}\,dV_0 = \int_{S^0} \vec{p}^0\cdot\delta\vec{v}\,dS_0 + \int_{V^0} \vec{f}^0\cdot\delta\vec{v}\,dV_0. \tag{76a}$$

$$\int_{V^0} S : \delta\dot{E}\,dV^0 = \int_{S^0} \vec{p}^0\cdot\delta\vec{v}\,dS^0 + \int_{V^0} \vec{f}^0\cdot\delta\vec{v}\,dV^0. \tag{76b}$$

$$\int_{V^0} F\,S : \nabla^0\delta\vec{v}\,dV^0 = \int_{S^0} \vec{p}^0\cdot\delta\vec{v}\,dS^0 + \int_{V^0} \vec{f}^0\cdot\delta\vec{v}\,dV^0. \tag{76c}$$

$$\int_{V^0} P : \nabla^0\delta\vec{v}\,dV^0 = \int_{S^0} \vec{p}^0\cdot\delta\vec{v}\,dS^0 + \int_{V^0} \vec{f}^0\cdot\delta\vec{v}\,dV^0. \tag{76d}$$

$$\int_{V^0} S : \delta E\,dV^0 = \int_{S^0} \vec{p}^0\cdot\delta\vec{u}\,dS^0 + \int_{V^0} \vec{f}^0\cdot\delta\vec{u}\,dV^0. \tag{76e}$$

$$\delta E = \frac{1}{2}\left(\nabla^0\delta\vec{u}^T\,F + F^T\,\nabla^0\delta\vec{u}\right). \tag{76f}$$

4. O objektivnosti tenzorjev

4.1. Superpozicija transformacij

$$F = F_2\,F_1. \tag{77a}$$

$$\begin{aligned} R^+\,U^+ = F^+ &= Q\,F = Q\,R\,U,\\ R^+ &= Q\,R,\\ U^+ &= U. \end{aligned} \tag{77b}$$

$$\begin{aligned} V^+\,R^+ = F^+ &= Q\,F = Q\,V\,R,\\ R^+ &= Q\,R,\\ V^+ &= Q\,V\,Q^T. \end{aligned} \tag{77c}$$

4.2. Dva opazovalca

4.3. Objektivni odvodi napetosti po času

4.3.1. Korotacijski ali Jaumannov odvod

Spin

$$\begin{aligned} W^+ &= \Omega + Q\,W\,Q^T\\ &= \dot{Q}\,Q^T + Q\,W\,Q^T. \end{aligned}$$

Odvod vektorja. Korotacijski odvod

$$\begin{aligned} \dot{u^+} - W^+\,u^+ &= (\dot{u^+}) - W^+\,u^+\\ &= \dot{Q}\,u + Q\,\dot{u} - \dot{Q}\,u - Q\,W\,u\\ &= (\dot{u} - W\,u)^+ = (\mathring{u})^+\\ &= Q\,(\dot{u} - W\,u) = Q\,\mathring{u}\,. \end{aligned}$$

Korotacijski odvod

$$\begin{aligned} \mathring{u} &= \dot{u} - Q\,u,\\ (\mathring{u})^+ &= Q\,\mathring{u}\,. \end{aligned}$$

Odvod tenzorja 2 reda. Iz enačb

$$\begin{aligned} A^+ &= Q\,A\,Q^T,\\ \dot{A^+} &= \dot{Q}\,A\,Q^T + Q\,\dot{A}\,Q^T + Q\,A\,\dot{Q}^T,\\ W^+ &= \Omega + Q\,W\,Q^T\\ &= \dot{Q}\,Q^T + Q\,W\,Q^T\\ W^T\,Q &= \dot{Q} + Q\,W\\ \dot{Q} &= W^T\,Q - Q\,W \end{aligned}$$

dobimo Jaumannov odvod

$$\begin{aligned} (\dot{A} - W\,A + A\,W)^+ &= Q\,(\dot{A} - W\,A + A\,W)\,Q^T,\\ \mathring{A} &= \dot{A} - W\,A + A\,W,\\ (\mathring{A})^+ &= Q\,\mathring{A}\,Q^T. \end{aligned}$$

Lastnosti Jaumannovega odvoda

$$2\,A : \mathring{A} = 2\,A : \dot{A} - 2\,A : (W\,A) + 2\,A : (A\,W) = 2\,A : \dot{A}.$$

4.3.2. Konvekcijski odvod

$$L^+ = \Omega + Q\,L\,Q^T.$$

Konvekcijski odvod

$$\overset{\triangle}{u} = \dot{u} + L^T\,u.$$

Odvod vektorja.

Odvod tenzorja 2 reda.

$$\overset{\triangle}{A} = \dot{A} + L^T A + A L$$

$$(\overset{\triangle}{A})^+ = Q \overset{\triangle}{A} Q^T.$$

4.4. Objektivni časovni odvodi tenzorjev napetosti

4.4.1. Jaumannov odvod

$$\overset{\nabla}{\sigma} = \dot{\sigma} - W\sigma + \sigma W. \tag{78}$$

4.4.2. Truesdellov odvod

$$\overset{\circ}{\sigma} = \dot{\sigma} - L\sigma - \sigma L^T + \mathrm{tr}(L)\,\sigma. \tag{79}$$

4.4.3. Green–Naghdijev odvod

$$\overset{\triangle}{\sigma} = \dot{\sigma} - \dot{R}R^T\sigma + \sigma\dot{R}R^T. \tag{80}$$

4.4.4. Oldroydov odvod

$$\overset{\circ}{\sigma} = \dot{\sigma} - L\sigma - \sigma L^T. \tag{81}$$

5. Enačbe snovi (konstitucijske enačbe)

5.1. Osnovni pojmi

5.1.1. Princip virtualnih hitrosti

Moč notranjih sil določajo izrazi:

$$\begin{aligned} &\int_{V^0} P : \dot{F}\, dV^0 \\ &\int_{V^0} S : \dot{E}\, dV^0 \\ &\int_{V} \sigma : D\, dV \end{aligned} \tag{82}$$

5.1.2. Zveze med deformacijami in napetostmi

Iz gornjih enač sledi, da so naravne možnosti za zveze med deformacijami in napetostmi takšne:

$$\begin{aligned} P &= f_P(F) \\ S &= f_S(E) \\ \sigma &= f_\sigma(D) \end{aligned} \tag{83}$$

Kadar je material nehomogen, so zveze lahko neposredno odvisne tudi od pozicije delca na telesu:

$$\begin{aligned} P &= f_P(F(\vec{r}^0), \vec{r}^0) \\ S &= f_S(E(\vec{r}^0), \vec{r}^0) \\ \sigma &= f_\sigma(D(\vec{r}), r) \end{aligned} \tag{84}$$

Zakoni so podani z enoličnimi, vendar v splošnem nelinearnimi funkcijami.

5.1.3. Specifična deformacijska energija $\mathscr{D}$

Specifična deformacijska energija

$$\mathcal{D} = \int_{t_1}^{t_2} P(f, \vec{r}) : \dot{F}\, dt \tag{85}$$

je v splošnem odvisna od obtežne poti delca v časovnem intervalu $[t_1, t_2]$, t.j. od spreminjanja deformacijskega gradienta na tem intervalu. V tem primeru model materiala imenujemo Cauchyjev model nelinearne elastičnosti.

5.2. Hiperelastični material

Specifična deformacijska energija je odvisna samo od začetnega in končnega stanja deformacij pri t_1 in t_2. To pa je mogoče samo, če je odvod deformacijske energije po času njen popoln diferencial

$$\dot{\mathcal{D}} = P : \dot{F} = \frac{\partial \mathcal{D}}{\partial F} : \dot{F}. \tag{86}$$

Od tu dobimo

$$\mathcal{D} = \int_{t_1}^{t_2} P(f, \vec{r}) : \dot{F}\, dt = \int_{t_1}^{t_2} \frac{\partial \mathcal{D}}{\partial t}\, dt = \mathcal{D}(t_2) - \mathcal{D}(t_1). \tag{87}$$

Iz enačbe (86) sledi zveza

$$\left(P - \frac{\partial \mathcal{D}}{\partial F}\right) : \dot{F} = 0 \tag{88}$$

Ker je po predpostavki P funkcija F in $\vec{r}^0$, velja

$$\mathcal{D} = \mathcal{D}(F, \vec{r}^0). \tag{89}$$

5.2.1. Objektivnost deformacijske energije

V delcu $\vec{r}^0$ se ob deformaciji telesa nakopiči spec. deformacijska energija $\mathcal{D}(F)$. Na to stanje superponiramo togo rotacijo $Q(t)$. Pri tem se F spremeni v $F^+ = Q F$. Zato se tudi $\mathcal{D}$ spremeni v $\mathcal{D}^+ = \mathcal{D}(Q F) \neq \mathcal{D}(F)$. To pa je kršitev objektivnosti, saj se deformacijska energija pri togem zasuku ne sme spremeniti.

Rešitev te zadrege je preprosta. Ker je po polarnem razcepu $F = R U$ je vse deformiranje vsebovano že v raztegu U, ki pa je hkrati neodvisen od dodanega togega zasuka. Dokazali smo zvezo $U^+ = U$. Običajno namesto U uporabimo kvadrat $C = U^2$ ali Green–Lagrangev tenzor deformacij $E = \frac{1}{2}(C - I)$.

Specifično deformacijsko energijo lahko opišemo z različnimi funkcijami

$$\mathcal{D}_F(F) = \mathcal{D}_U(U) = \mathcal{D}_C(C) = \mathcal{D}_E(E). \tag{90}$$

Od tega so samo zadnji trije opisi objektivni. V nadaljevanju različnih oznak za $\mathcal{D}$ ne bomo uporabljali.

5.2.2. Določitev PK2 iz $\mathcal{D}(E)$ in $\mathcal{D}(C)$

Upoštevamo zvezo

$$P : \dot{F} = S : \dot{E} \tag{91}$$

ter

$$\dot{\mathcal{D}}_F = P : \dot{F} = S : \dot{E} = \dot{\mathcal{D}}_E = \dot{\mathcal{D}} = \frac{\partial \mathcal{D}}{\partial E} : \dot{E} \tag{92}$$

Iz gornje enačbe neposredno dobimo

$$S = \frac{\partial \mathcal{D}}{\partial E}. \tag{93}$$

Upoštevamo $\dot{E} = \frac{1}{2}\,\dot{C}$ in dobimo še

$$\dot{\mathcal{D}} = \frac{1}{2}\,S : \dot{C} = S : \dot{E} = \frac{\partial \mathcal{D}}{\partial C} : \dot{C} = 2\,\frac{\partial \mathcal{D}}{\partial C} : \dot{E} \tag{94}$$

in od tu

$$S = 2\,\frac{\partial \mathcal{D}}{\partial C} \tag{95}$$

$$\frac{\partial D}{\partial U} = \frac{\partial D}{\partial C}\,\frac{\partial C}{\partial U} = \frac{\partial D}{\partial C}\,\frac{\partial U^2}{\partial U} = 2\,\frac{\partial D}{\partial U}\,U \tag{96}$$

5.2.3. St. Venant–Kirchhoffov material

Specifična deformacijska energija materiala je

$$\mathcal{D}(E) = \frac{1}{2}\,\lambda\,(\mathrm{tr}(E))^2 + \mu\,E : E, \tag{97}$$

kjer sta E in μ materialna parametra. Tenzor S dobimo z odvajanjem

$$\begin{aligned} S &= \frac{\partial \mathcal{D}(E)}{\partial E} \\ &= \frac{1}{2}\,2\,\lambda\,(\mathrm{tr}(E))\,\frac{\partial(\mathrm{tr}(E))}{\partial E} + \mu\left(\frac{\partial E}{\partial E} : E + E : \frac{\partial E}{\partial E}\right) \\ &= \lambda\,(\mathrm{tr}E)\,I + 2\,\mu\,E. \end{aligned} \tag{98}$$

ali v komponentni obliki

$$S_{ij} = \lambda\,E_{kk}\,\delta_{ij} + 2\,\mu\,E_{ij} = C_{ijkl}\,E_{kl}. \tag{99}$$

in od tu izpeljemo konstitucijski tenzor C s komponentami

$$C_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + 2\,\mu\,\delta_{ik}\,\delta_{jl}. \tag{100}$$

Pri izpeljavi gornje enačbe smo uporabili zveze

$$\frac{\partial \mathrm{tr}(A))}{\partial A} = \frac{\partial(A_{kk})}{\partial A_{ij}}\,\vec{e}_i \otimes \vec{e}_j = \delta_{ki}\,\delta_{kj}\,\vec{e}_i \otimes \vec{e}_j = \delta_{ij}\,\vec{e}_i \otimes \vec{e}_j = I_2 \tag{101}$$

$$\frac{\partial A}{\partial A} = \frac{\partial A_{ij}}{\partial A_{kl}}\,\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l = \delta_{ij}\,\delta_{kl}\,\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l = I_4. \tag{102}$$

$$\begin{aligned} I_4 : A &= \delta_{ij}\,\delta_{kl}\,\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l : A_{mn}\,\vec{e}_m \otimes \vec{e}_n \\ &= \delta_{ij}\,\delta_{kl}\,A_{mn}\,\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}_l : \vec{e}_m \otimes \vec{e}_n \\ &= \delta_{ij}\,\delta_{kl}\,A_{mn}\,(\vec{e}_i \otimes \vec{e}_j)\,(\vec{e}_k \cdot \vec{e}_m)\,(\vec{e}_l \cdot \vec{e}_n) \\ &= \delta_{ij}\,\delta_{kl}\,A_{mn}\,(\vec{e}_i \otimes \vec{e}_j)\,\delta_{km}\,\delta_{ln} = A_{kl}\,(\vec{e}_k \otimes \vec{e}_l) \\ &= A. \end{aligned} \tag{103}$$

$$\vec{a} \otimes \vec{b} \otimes \vec{c} \otimes \vec{d} : \vec{e} \otimes \vec{f} = (\vec{a} \otimes \vec{b})\,(\vec{c} \cdot \vec{e})\,(\vec{d} \cdot \vec{f}) \tag{104}$$

Podobno dobimo $A : I_4 = A$, pri čemer uporabimo pravilo

$$\vec{a} \otimes \vec{b} : \vec{c} \otimes \vec{d} \otimes \vec{e} \otimes \vec{f} = (\vec{e} \otimes \vec{f})\,(\vec{a} \cdot \vec{c})\,(\vec{b} \cdot \vec{d}) \tag{105}$$

Z uporabo enačb

$$\sigma = J^1\,F\,S\,F^T \tag{106}$$

in

$$P = J\,\sigma\,F^{-T} = F\,S \tag{107}$$

izračunamo še komponente Cauchyevega in PK1 tenzorja

$$\sigma_{ij} = \frac{1}{J}\,F_{ik}\,S_{km}\,F^T_{mj} = \frac{1}{J}\,F_{ik}\,S_{km}\,F_{jm} \tag{108}$$

$$P_{ij} = F_{ik}\,S_{kj} \tag{109}$$

5.2.4. Objektivnost hiperelastičnih zakonov

Materialni opis:

$$S^+ = \frac{\partial \mathcal{D}(E^+)}{\partial E^+} = \frac{\partial \mathcal{D}(E)}{\partial E} = S. \tag{110}$$

Prostorski opis:

$$\begin{aligned} \sigma^+ &= (J^+)^{-1}\,(F^+)\,(S^+)\,(F^+)^T \\ &= (J)^{-1}\,(Q\,F)\,(S)\,(Q\,F)^T = Q\,\sigma\,Q^T. \end{aligned} \tag{111}$$

5.3. Hiperelastični material z vezmi

Pri običajnem hiperelastičnem materialu je $\dot{C}$ poljuben tenzor z mendsebojno neodvisnimi komponentami zato lahko povzamemo

$$\left(\frac{1}{2}\,S - \frac{\partial \mathcal{D}}{\partial C}\right) : \dot{C} = 0 \longrightarrow \frac{1}{2}\,S = \frac{\partial \mathcal{D}}{\partial C}. \tag{112}$$

Pri materialih z vezjo pa komponente $\dot{C}$ niso več neodvisne in gornji sklep zato ne velja. Zaenkrat obravnavajmo skalarno vez oblike

$$\varphi : C \mapsto \varphi(C) = 0. \tag{113}$$

Vez odvajamo po času in dobimo

$$\dot{\varphi} = \frac{\partial \mathcal{D}}{\partial C} : \dot{C} = 0. \tag{114}$$

Gornjo enačbo lahko pomnožimo s poljubnim skalarjem $\frac{1}{2}\,\alpha = \frac{1}{2}\,\alpha(t)$ in dobimo

$$\frac{1}{2}\,\alpha\,\dot{\varphi} = \frac{1}{2}\,\alpha\,\frac{\partial \varphi}{\partial C} : \dot{C} = 0. \tag{115}$$

Iz enačb

$$\begin{aligned} \left(\frac{1}{2}\,S - \frac{\partial \mathcal{D}}{\partial C}\right) : \dot{C} &= 0 \\ \frac{1}{2}\,\alpha\,\frac{\partial \varphi}{\partial C} : \dot{C} &= 0 \end{aligned} \tag{116}$$

dobimo

$$\left(\frac{1}{2}\,S - \frac{\partial \mathcal{D}}{\partial C}\right) = \frac{1}{2}\,\alpha\,\frac{\partial \varphi}{\partial C} \tag{117}$$

Podobno postopamo v primeru večjega števila vezi:

$$\begin{aligned} \varphi_1 &: C \mapsto \varphi_1(C) = 0, \\ &\vdots \\ \varphi_n &: C \mapsto \varphi_n(C) = 0. \end{aligned} \tag{118}$$

Ker ima C šest komponent, je vezi lahko največ 6. Podobno kot v primeru ene vezi izpeljemo enakosti

$$\begin{aligned} \left(\frac{1}{2}\,S - \frac{\partial \mathcal{D}}{\partial C}\right) : \dot{C} &= 0 \\ \frac{1}{2}\,\alpha_1\,\frac{\partial \varphi_1}{\partial C} : \dot{C} &= 0, \\ &\vdots \\ \frac{1}{2}\,\alpha_n\,\frac{\partial \varphi_n}{\partial C} : \dot{C} &= 0. \end{aligned} \tag{119}$$

in od tu neposredno dobimo

$$\left(\frac{1}{2}\,S - \frac{\partial \mathcal{D}}{\partial C}\right) = \sum_{i=1}^{n} \frac{1}{2}\,\alpha_i\,\frac{\partial \varphi_i}{\partial C} \tag{120}$$

oziroma

$$S = 2\,\frac{\partial \mathcal{D}}{\partial C} + \sum_{i=1}^{n} \alpha_i\,\frac{\partial \varphi_i}{\partial C} \tag{121}$$

5.3.1. Nestisljiv material

V tem primeru se vezna enačba v krepki obliki glasi

$$\varphi : J \mapsto J - 1 = 0, \tag{122}$$

v šibki obliki pa

$$\dot{\varphi} = \dot{J} = \frac{\partial J}{\partial C} : \dot{C} = \frac{1}{2}\,J\,C^{-1} : \dot{C}, \tag{123}$$

kjer smo upoštevali zvezo

$$\frac{\partial J}{\partial C} = \frac{1}{2}\,J\,C^{-1}. \tag{124}$$

Z upoštevanjem gornje enačbe dobimo

$$S = 2\,\frac{\partial \mathcal{D}}{\partial C} + \frac{1}{2}\,\alpha\,J\,C^{-1} \tag{125}$$

Parameter α v gorni enačbi moremo povezati s hidrostatično napetostjo. Cauchyev tenzor napetosti σ lahko razstavimo na hidrostatični in deviatorični del

$$\sigma = \sigma' + p\,I. \tag{126}$$

Tenzor PK2 potem lahko zapišemo v obliki

$$\begin{aligned} S &= J\,F^{-1}\,\sigma\,F^{-T} \\ &= J\,F^{-1}\,(\sigma' + p\,I)\,F^{-T} \\ &= J\,F^{-1}\,(\sigma')\,F^{-T} + J\,F^{-1}\,(p\,I)\,F^{-T} \\ &= S' + p\,J\,C^{-1} \end{aligned} \tag{127}$$

S' ima pomembno lastnost, da je $S' : C = 0$.

$$\begin{aligned} S' : C &= \mathrm{tr}(C^T\,S) = \mathrm{tr}(C\,S) \\ &= \mathrm{tr}(F^T\,F\,J F^{-1} \sigma'\,F^{-T}) \\ &= J\,\mathrm{tr}(F^T\,\sigma'\,F^{-T}) \\ &= J\,\mathrm{tr}(\sigma') = 0. \end{aligned} \tag{128}$$

Izračunamo vsoto

$$\begin{aligned} S : C &= S' : C + p\,J\,C^{-1} : C \\ &= 0 + p\,J\,\mathrm{tr}(C\,C^{-1}) \\ &= p\,J\,\mathrm{tr}(I) \\ &= 3\,p\,J \end{aligned} \tag{129}$$

in od tu

$$\begin{aligned} p &= \frac{1}{3}\,J^{-1}\,S : C \\ &= \frac{1}{3}\,J^{-1}\left(2\,\frac{\partial \mathcal{D}}{\partial C} + \frac{1}{2}\,\alpha\,J\,C^{-1}\right) : C \\ &= \frac{2}{3}\,J^{-1}\,\frac{\partial \mathcal{D}}{\partial C} : C + \frac{1}{6}\,\alpha\,C^{-1} : C \\ &= \frac{2}{3}\,J^{-1}\,\frac{\partial \mathcal{D}}{\partial C} : C + \frac{1}{2}\,\alpha \end{aligned} \tag{130}$$

Parameter $\frac{1}{2}\,\alpha$ bi torej predstavljal hidrostatično napetost p v primeru, ko bi bil prvi člen $\frac{2}{3}\,J^{-1}\,\frac{\partial \mathcal{D}}{\partial C} : C$ enak nič. Sedaj sem nam zastavi tole vprašanje. Kakšna naj bo fukcija $\mathcal{D}$ da bo izraz $\frac{\partial \mathcal{D}}{\partial C} : C$ enak nič? Odgovor na to vprašanje je takšen. Funkcijo $\mathcal{D}$ zamenjamo s funkcijo $\hat{\mathcal{D}} : C \mapsto \mathcal{D}(\hat{C}) = \mathcal{D}(J^{\frac{-2}{3}}\,C)$. Potem lahko pokažemo, da velja $\frac{\partial \hat{\mathcal{D}}}{\partial C} : C = 0$. V tem primeru velja $p = \frac{1}{2}\alpha$. Vse skupaj povzamemo v tale rezultat

$$\begin{aligned} S &= 2\,\frac{\partial \hat{\mathcal{D}}}{\partial C} + p\,J\,C^{-1}, \\ \hat{\mathcal{D}} &: C \mapsto \mathcal{D}(\hat{C}) = \mathcal{D}(J^{\frac{-2}{3}}\,C). \end{aligned} \tag{131}$$

5.3.2. Nestisljiv Neo Hookeov material

Deformacijsko energijo nestisljivega Neo Hookeovega materiala opišemo s funkcijo

$$\mathcal{D}(C) = \frac{1}{2}\,\mu\,(\mathrm{tr}(C) - 3). \tag{132}$$

Modificirana funkcija se potem glasi

$$\hat{\mathcal{D}}(C) = \frac{1}{2}\,\mu\,(\mathrm{tr}(J^{\frac{-2}{3}}\,C) - 3). \tag{133}$$

Potrebujemo odvod

$$
\begin{aligned}
\frac{\partial \hat{\mathcal{D}}(C)}{\partial C} &= \frac{1}{2}\,\mu\, \frac{\partial(\mathrm{tr}(J^{\frac{-2}{3}}\, C) - 3)}{\partial C} \\
&= \frac{1}{2}\,\mu\, \frac{\partial(J^{\frac{-2}{3}}\, C : I)}{\partial C} \\
&= \frac{1}{2}\,\mu \left(\frac{\partial J^{\frac{-2}{3}}}{\partial C}\, C : I + \frac{\partial(C : I)}{\partial C}\, J^{\frac{-2}{3}} \right) \\
&= \frac{1}{2}\,\mu \left(\frac{-2}{3}\, J^{\frac{-5}{3}}\, \frac{\partial J}{\partial C}\, C : I + I\, J^{\frac{-2}{3}} \right) \\
&= \frac{1}{2}\,\mu \left(\frac{-2}{3}\, J^{\frac{-5}{3}}\, \frac{1}{2}\, J\, C^{-1}\, C : I + I\, J^{\frac{-2}{3}} \right) \\
&= \frac{1}{2}\,\mu\, J^{\frac{-2}{3}} \left(I - \frac{1}{3}\, C^{-1}\, I_C \right),
\end{aligned}
\tag{134}
$$

kjer smo z I_C označili prvo invarianto od C. Od tu neposredno izpeljemo

$$
S = \mu\, J^{\frac{-2}{3}} \left(I - \frac{1}{3}\, C^{-1}\, I_C \right) + p\, J\, C^{-1}. \tag{135}
$$

Z uporabo enačbe $\sigma = J^{-1}\, F\, S\, F^T$ lahko po krajšem računu dobimo

$$
\begin{aligned}
\sigma &= \mu\, J^{\frac{-5}{3}}\, F \left(I - \frac{1}{3}\, C^{-1}\, I_C \right) F^T + p\, F\, C^{-1}\, F^T \\
&= \mu\, J^{\frac{-5}{3}}\, F \left(I - \frac{1}{3}\, F^{-1}\, F^{-T}\, I_C \right) F^T + p\, F\, F^{-1}\, F^{-T}\, F^T \\
&= \mu\, J^{\frac{-5}{3}} \left(B - \frac{1}{3}\, I\, I_B \right) + p\, I \\
&= \sigma' + p\, I.
\end{aligned}
\tag{136}
$$

kjer smo s

$$
\sigma' = \mu\, J^{\frac{-5}{3}} \left(B - \frac{1}{3}\, I\, I_B \right) \tag{137}
$$

označili deviator. Zapišimo PK2 in Cauchyev tenzor še po komponentah: Materialna oblika:

$$
S_{ij} = \mu\, J^{\frac{-2}{3}} \left(\delta_{ij} - \frac{1}{3}\, C^{-1}_{ij}\, I_C \right) + p\, J\, C^{-1}_{ij} \tag{138}
$$

Prostorska oblika:

$$
\sigma_{ij} = \mu\, J^{\frac{-5}{3}} \left(B_{ij} - \frac{1}{3}\, \delta_{ij}\, I_B \right) + p\, \delta_{ij}. \tag{139}
$$

5.4. *Izotropen hiperelastični material*

Izotropija materiala pomeni, da je njegovo obnašanje neodvisno od smeri materialnega vlakna. Posledica te zahteve je, da mora biti specifična deformacijska energija odvisna od količin, ki so neodvisne od smeri, to pa so npr. invariante tenzorja C.

$$
\begin{aligned}
I_C &= \mathrm{tr}(C) = C : I \\
II_C &= \mathrm{tr}(C\, C) = C : C \\
III_C &= \det(C) = \det(F^T\, F) = (\det(F))^2 = J^2
\end{aligned}
\tag{140}
$$

Če je material prost vezi je njegova deformacijska energija $\mathcal{D} = \mathcal{D}(I_C, II_C, III_C)$. Napetosti tenzor PK2 lahko potem izračunamo po enačbi

$$
\begin{aligned}
S &= 2\, \frac{\partial \mathcal{D}}{\partial C} \\
&= 2\, \frac{\partial \mathcal{D}}{\partial I_C}\, \frac{\partial I_C}{\partial C} + 2\, \frac{\partial \mathcal{D}}{\partial II_C}\, \frac{\partial II_C}{\partial C} + 2\, \frac{\partial \mathcal{D}}{\partial III_C}\, \frac{\partial III_C}{\partial C} \\
&= 2\, \frac{\partial \mathcal{D}}{\partial I_C}\, I + 4\, \frac{\partial \mathcal{D}}{\partial II_C}\, C + 2\, \frac{\partial \mathcal{D}}{\partial III_C}\, J^2\, C^{-1},
\end{aligned}
\tag{141}
$$

kjer smo upoštevali zveze

$$
\begin{aligned}
\frac{\partial I_C}{\partial C} &= I, \\
\frac{\partial I_C}{\partial C} &= 2\, C, \\
\frac{\partial I_C}{\partial C} &= J^2\, C^{-1}.
\end{aligned}
\tag{142}
$$

Z upoštevanjem enačbe $\sigma = J^{-1}\, F\, S\, F^T$ in enakosti

$$
\begin{aligned}
I_C &= I_B, \\
II_C &= II_B, \\
III_C &= III_B, \\
\mathcal{D}(I_C, II_C, III_C) &= \mathcal{D}(I_B, II_B, III_B)
\end{aligned}
\tag{143}
$$

moremo tudi tenzor σ po krajšem izračunu izraziti z odvodi invariant

$$
\sigma = 2\, J^{-1}\, \frac{\partial \mathcal{D}}{\partial I_B}\, B + 4\, J^{-1}\, \frac{\partial \mathcal{D}}{\partial II_B}\, B^2 + 2\, J\, \frac{\partial \mathcal{D}}{\partial III_B}\, I. \tag{144}
$$

5.4.1. *Stisljiv izotropen Neo Hookeov material*

Deformacijsko energijo stisljivega izotropnega neo Hookeovega materiala podaja enačba

$$
\mathcal{D} = \frac{1}{2}\, \mu\, (I_C - 3) - \mu\, \ln J + \frac{1}{2}\, \lambda\, (\ln J)^2, \tag{145}
$$

kjer sta λ in μ materialni konstanti, determinanta $J = \sqrt{III_C}$. Deformacijska energija je torej funkcija prve in tretje invariante C. Izračunamo odvode

$$
\begin{aligned}
\frac{\partial \mathcal{D}}{\partial I_C} &= \frac{1}{2}\, \mu, \\
\frac{\partial \mathcal{D}}{\partial II_C} &= 0, \\
\frac{\partial \mathcal{D}}{\partial III_C} &= \frac{1}{2\, J^2}(-\mu + \lambda\, \ln J)
\end{aligned}
\tag{146}
$$

in napetostna tenzorja

$$
\begin{aligned}
S &= 2\, \frac{\partial \mathcal{D}}{\partial I_C}\, I + 4\, \frac{\partial \mathcal{D}}{\partial II_C}\, C + 2\, \frac{\partial \mathcal{D}}{\partial III_C}\, J^2\, C^{-1} \\
&= \mu\, (I - C^{-1}) + \lambda\, (\ln J)\, C^{-1}
\end{aligned}
\tag{147}
$$

in

$$
\begin{aligned}
\sigma &= 2\, J^{-1}\, \frac{\partial \mathcal{D}}{\partial I_B}\, B + 4\, J^{-1}\, \frac{\partial \mathcal{D}}{\partial II_B}\, B^2 + 2\, J\, \frac{\partial \mathcal{D}}{\partial III_B}\, I \\
&= \frac{\mu}{J}(B - I) + \frac{\lambda}{J}(\ln J)\, I.
\end{aligned}
\tag{148}
$$

Še v komponetni obliki

$$S_{ij} = \mu\,\delta_{ij} + (\lambda\,\ln J - \mu)\,C^{-1}_{ij}, \tag{149}$$

$$\sigma_{ij} = \frac{\mu}{J}\,B_{ij} + \frac{1}{J}(\lambda\,\ln J - \mu)\,\delta_{ij}. \tag{150}$$

5.4.2. Izotropen hiperelastični material z aditivno specifično deformacijsko energijo

Pri nekaterih materialih moremo ločeno obravnavati vpliv volumskih in distorzijskih deformacij. V takšnih primerih lahko specifično deformacijsko energijo zapišemo z vsoto specifične deformacijske energije zaradi distorzijskih in specifične deformacijske energije zaradi volumskih deformacij.

$$\mathcal{D}(C) = \hat{\mathcal{D}}(C) + U(J), \tag{151}$$

kjer je $\hat{\mathcal{D}}(C) = \mathcal{D}(J^{\frac{-2}{3}}\,C)$, $U(J)$ pa specifična deformacijska energija zaradi volumske spremembe $J = \det F$. Takšna funkcija je npr.

$$U(J) = \frac{1}{2}\kappa\,(J-1)^2. \tag{152}$$

če je $J = 1$, se volumen ni spremenil in je zato pripadajoča deformacijska energija enaka nič ($U(1) = 0$). Sicer pa je v izbranem primeru kvadratična funkcija od $J-1$. Konkretno funkcijo $U(J)$ moramo v splošnem dobiti za vsak material posebej s preizkusi, v katerih se preizkušancu spreminja samo volumen. Tenzor PK2 izračunamo po enačbi

$$\begin{aligned} S &= 2\,\frac{\partial\mathcal{D}}{\partial C} \\ &= 2\,\frac{\partial\hat{\mathcal{D}}}{\partial C} + 2\,\frac{\partial U}{\partial C} \\ &= 2\,\frac{\partial\hat{\mathcal{D}}}{\partial C} + 2\,\frac{\partial U}{\partial J}\,\frac{\partial J}{\partial C} \\ &= 2\,\frac{\partial\hat{\mathcal{D}}}{\partial C} + p\,J\,C^{-1} \\ &= 2\,\frac{\partial\hat{\mathcal{D}}}{\partial C} + \kappa\,(J-1)\,J\,C^{-1}, \end{aligned} \tag{153}$$

kjer smo ponovno uporabili zvezo $\frac{\partial J}{\partial C} = \frac{1}{2}\,J\,C^{-1}$ in iz primerjave z enačbo (**??**) še zvezo $\frac{\partial U}{\partial J} = p$. Z upoštevanjem zveze $\sigma = J^{-1}\,F\,S\,F^T$ izračunamo še Cauchyjev tenzor napetosti

$$\sigma = 2\,J^{-1}\,F\,\frac{\partial\hat{\mathcal{D}}}{\partial C}\,F^T + \kappa\,(J-1)\,I. \tag{154}$$

Kombinacija z nestisljivim neo Hookeovim materialom. Uporabimo rezultat enačbe (**??**)

$$\frac{\partial\hat{\mathcal{D}}(C)}{\partial C} = \frac{1}{2}\,\mu\,J^{\frac{-2}{3}}\left(I - \frac{1}{3}\,C^{-1}\,I_C\right) \tag{155}$$

in dobimo

$$S = \mu\,J^{\frac{-2}{3}}\left(I - \frac{1}{3}\,C^{-1}\,I_C\right) + \kappa\,(J-1)\,J\,C^{-1}, \tag{156}$$

$$\sigma = \mu\,J^{\frac{-5}{3}}\left(B - \frac{1}{3}\,I\,I_B\right) + \kappa\,(J-1)\,I. \tag{157}$$

5.4.3. Izotropen hiperelastični material izražen z raztegi

Tako kot invariante Cauchy–Greenovega tenzorja C so tudi kvadrati raztegov λ_1^2, λ_2^2, λ_3^2, to je lastne vrednosti tenzorja C neodvisne od izbire koordinatega sistema. Zato lahko specifično deformacijsko energijo izrazimo z raztegi λ_1, λ_2 in λ_3 kar zapišemo z enačbo

$$\mathcal{D} = \mathcal{D}(\lambda_1, \lambda_2, \lambda_3). \tag{158}$$

Napetostni tenzor PK2 izrazimo po enačbi

$$S = 2\,\frac{\partial\mathcal{D}(C)}{\partial C} = 2\left(\frac{\partial\mathcal{D}}{\partial\lambda_1}\,\frac{\partial\lambda_1}{\partial C} + \frac{\partial\mathcal{D}}{\partial\lambda_2}\,\frac{\partial\lambda_2}{\partial C} + \frac{\partial\mathcal{D}}{\partial\lambda_3}\,\frac{\partial\lambda_3}{\partial C}\right). \tag{159}$$

Upoštevamo zveze

$$\begin{aligned} \frac{\partial\lambda_1}{\partial C} &= \frac{1}{2\,\lambda_1}\,(\vec{m}_1^0 \otimes \vec{m}_1^0), \\ \frac{\partial\lambda_2}{\partial C} &= \frac{1}{2\,\lambda_2}\,(\vec{m}_2^0 \otimes \vec{m}_2^0), \\ \frac{\partial\lambda_3}{\partial C} &= \frac{1}{2\,\lambda_3}\,(\vec{m}_3^0 \otimes \vec{m}_3^0). \end{aligned} \tag{160}$$

izpeljane v dodatku in povzamemo

$$S = \sum_{i=1}^{3}\frac{\partial\mathcal{D}}{\partial\lambda_i}\,\frac{1}{\lambda_i}\,(\vec{m}_i^0 \otimes \vec{m}_i^0). \tag{161}$$

Iz tega rezultata izluščimo glavne napetosti $S_{\alpha\alpha}$

$$S_{\alpha\alpha} = \frac{\partial\mathcal{D}}{\partial\lambda_\alpha}\,\frac{1}{\lambda_\alpha}, \quad \alpha = 1, 2, 3. \tag{162}$$

Določimo še Cauchyjev tenzor napetosti σ.

$$\begin{aligned} \sigma &= J^{-1}\,F\,S\,F^T \\ &= J^{-1}\sum_{i=1}^{3}\frac{1}{\lambda_i}\,\frac{\partial\mathcal{D}}{\partial\lambda_i}\,F\,(\vec{m}_i^0 \otimes \vec{m}_i^0)\,F^T \\ &= \sum_{i=1}^{3}\frac{1}{J\,\lambda_i}\,\frac{\partial\mathcal{D}}{\partial\lambda_i}\,(F\,\vec{m}_i^0) \otimes (F\,\vec{m}_i^0) \\ &= \sum_{i=1}^{3}\frac{\lambda_i}{J}\,\frac{\partial\mathcal{D}}{\partial\lambda_i}\,(\vec{m}_i \otimes \vec{m}_i) \\ &= \sum_{i=1}^{3}\frac{1}{J}\,\frac{\partial\mathcal{D}}{\partial(\ln\lambda_i)}\,(\vec{m}_i \otimes \vec{m}_i). \end{aligned} \tag{163}$$

Pri izpeljavi gornje enačbe smo upoštevali kinematične zveze

$$F\,\vec{m}_\alpha^0 = \lambda_\alpha\,\vec{m}_\alpha, \quad \alpha = 1, 2, 3. \tag{164}$$

Od tu podobno kot v prvem primeru povzamemo glavne normalne napetosti

$$\sigma_{\alpha\alpha} = \frac{1}{J}\,\frac{\partial\mathcal{D}}{\partial(\ln\lambda_\alpha)}, \quad \alpha = 1, 2, 3. \tag{165}$$

6. Linearizacija

[3]

6.1. Hiperelastični material z vezmi

Literatura

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