3. Vaja: tenzorji deformacij, fizikalni pomen komponent tenzorjev deformacij, specifična sprememba dolžine, sprememba pravega kota, povezava s predmetom Trdnost

Rado Flajs

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1. Naloga

1.1. Naloga

1.1.1. Polje pomikov materialnem opisu

Deformiranje tanke stene je opisano s poljem pomikov $\mathbf{u}$ v telesnih koordinatah $x \equiv x_0^1$, $y \equiv x_0^2$, $z \equiv x_0^3$:

(a) $\mathbf{u}(x,y,z) = u_x\mathbf{e}_x + u_y\mathbf{e}_y = a\,y\,\mathbf{e}_x + a\,x\,\mathbf{e}_y \Longrightarrow u_x = a\,y,\ u_y = a\,x,$

(b) $\mathbf{u}(x,y,z) = u_x\mathbf{e}_x + u_y\mathbf{e}_y = -a\,y\,\mathbf{e}_x + a\,x\,\mathbf{e}_y \Longrightarrow u_x = -a\,y,\ u_y = a\,x,$

(c) $\mathbf{u}(x,y,z) =$ rotacija okrog osi z za kot α,
$\mathbf{u}(x,y,z) = ((\cos\alpha - 1)\,x - \sin\alpha\,y)\mathbf{e}_x + ((\cos\alpha - 1)\,y + \sin\alpha\,x)\mathbf{e}_y$

(d) $\mathbf{u}(x,y,z) = 10^{-4}(2\,x^2\,\mathbf{e}_x - (x+y)^2\,\mathbf{e}_y + 4\,\mathbf{e}z),$

Podatki: $a = 10^{-4}$, $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$, $D(1,1,0)$ in $E(0.5,0.5,0)$, $T_1(10,10,0)$, $T_2(11,11,0)$. Vse razdalje so v metrih.

Deformiranje tanke stene je podajajo zveze med prostorskimi in telesnimi koordinatami delcev:

(e) $x_1 = \sqrt{3}\,x_1^0 + x_2^0, \quad x_2 = 2\,x_2^0, \quad x_3 = x_3^0.$

1.1.2. Naloga 1

V primerih (a), (b), (c) določi:

(a) polje pomikov v prostorskem opisu

(b) deformacijski gradient F

(c) komponente tenzorja majhnih deformacij ε_{ij}, komponente Green Lagrangevega tenzorja velikih deformacij E_{ij}, komponente tenzorja rotacij ω_{ij} v kartezičnem koordinatnem sistemu in vektor zasuka $\boldsymbol{\omega}$,

(d) točne vrednosti specifični sprememb dolžin daljic AB, AC, AD in BC in sprememb pravih kotov CAB in CED,

(e) deformirane bazne vektorje $\mathbf{g}_i = F\mathbf{e}_i$.

(f) specifično spremembo površine

(g) glavne normalne deformacije in pripadajoče smeri.

Email address: `rado.flajs@fgg.uni-lj.si` (Rado Flajs)

1.1.3. Naloga 2

V primeru (d) določi:

(a) spremembo razdalje med delcema $\mathcal{D}_1$ in $\mathcal{D}_2$, ki sta v nedeformiranem stanju določena s točkama $T_1(10, 10, 0)$ in $T_2(11, 11, 0)$.

(b) V tocki T_1 določi točno vrednost specifične spremembe dolžine v smeri T_1T_2. Kolikšno napako narediš, če to specifično spremembo dolžine izraziš

  (1) z vrednostjo tenzorja majhnih deformacij v smeri T_1T_2,

  (2) s povprečno vrednostjo specifične spremembe dolžine med točkama T_1 in T_2?

(c) Določi velikosti in smeri glavnih in normalnih kotnih deformacij v točki T_1.

1.1.4. Naloga 3

V primeru (e) določi:

(a) deformacijski gradient F,

(b) levi Cauchyev tenzor deformacij C,

(c) desni Cauchyev tenzor deformacij B,

(d) Green Lagrangev tenzor velikih deformacij E,

(e) Euler Almansijev tenzor velikih deformacij e,

(f) polarni razcep deformacijskega gradienta F,

(g) fizikalno pojasni dobljene rezultate.

1.2. Rešitev

1.2.1. Polje pomikov v primeru (c)

Z upoštevanjem enačbe

$$\boldsymbol{u} = \sin\omega\, \boldsymbol{e}_\omega \times \boldsymbol{r} + (1 - \cos\omega)\, \boldsymbol{e}_\omega \times (\boldsymbol{e}_\omega \times \boldsymbol{r})$$

in podatkov

$$\omega = \alpha, \quad \boldsymbol{e}_\omega = \boldsymbol{e}_z, \quad \boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y,$$
$$\boldsymbol{e}_z \times \boldsymbol{r} = x\boldsymbol{e}_y - y\boldsymbol{e}_x, \quad \boldsymbol{e}_z \times (\boldsymbol{e}_z \times \boldsymbol{r}) = -x\boldsymbol{e}_x - y\boldsymbol{e}_y,$$

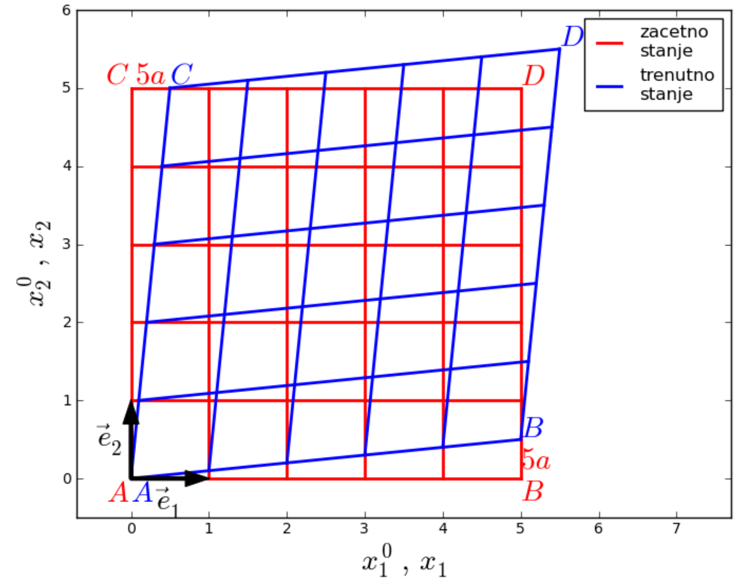
lahko pomik zapišemo z enačbo

$$\begin{aligned}\boldsymbol{u} &= \sin\alpha\, \boldsymbol{e}_z \times \boldsymbol{r} + (1 - \cos\alpha)\, \boldsymbol{e}_z \times (\boldsymbol{e}_z \times \boldsymbol{r}),\\ &= (-x(1 - \cos\alpha) - y\sin\alpha)\boldsymbol{e}_x + (x\sin\alpha - y(1 - \cos\alpha))\boldsymbol{e}_y\end{aligned}$$

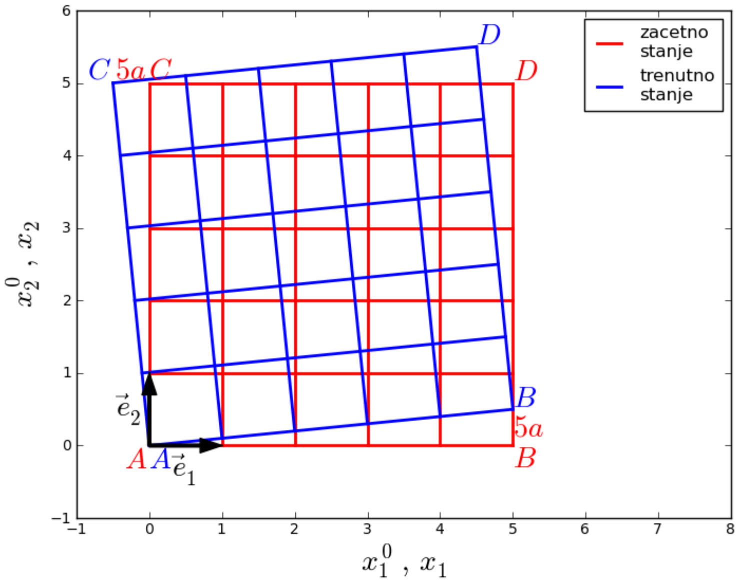
oziroma po komponentah

$$u_x = -x(1 - \cos\alpha) - y\,\sin\alpha, \quad u_y = x\,\sin\alpha - y(1 - \cos\alpha).$$

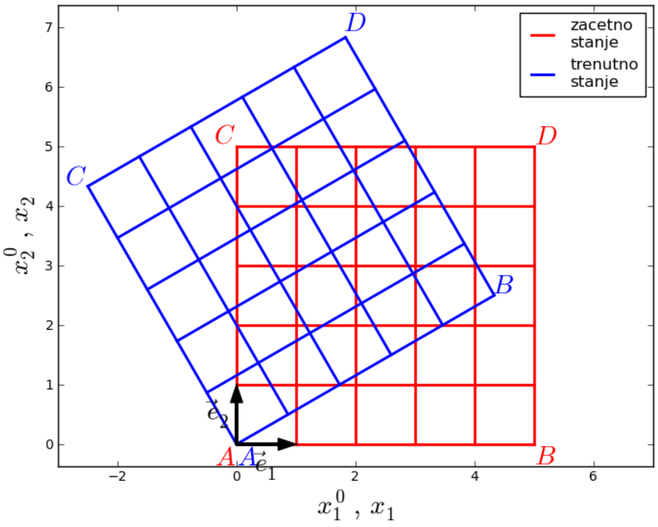
1.2.2. *Primer (a):* $\mathbf{u}(x, y, z) = a\,y\,\mathbf{e}_x + a\,x\,\mathbf{e}_y$.



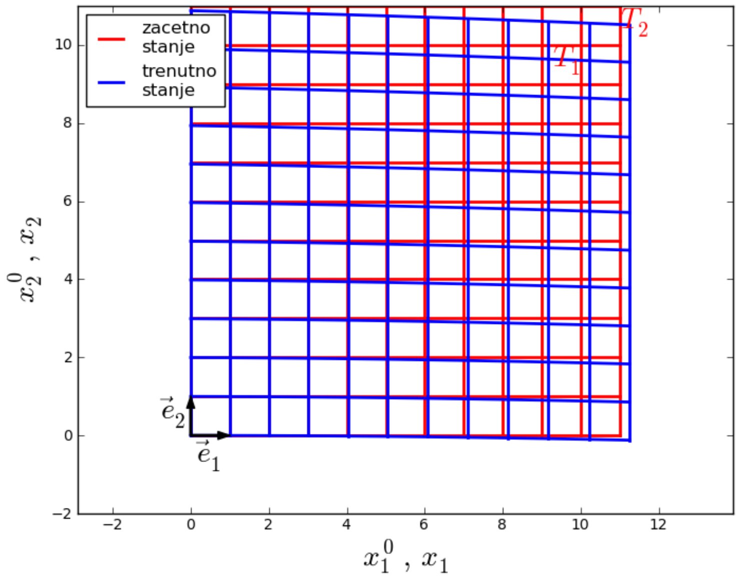
1.2.3. *Primer (b):* $\mathbf{u}(x, y, z) = -a\,y\,\mathbf{e}_x + a\,x\,\mathbf{e}_y$.



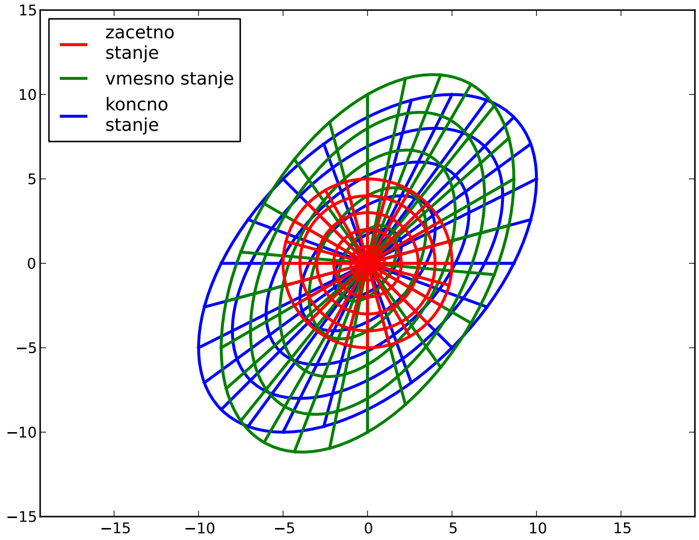
1.2.4. Primer (c): $\mathbf{u}(x, y, z) = ((\cos\alpha - 1)\, x - \sin\alpha\, y)\mathbf{e}_x + ((\cos\alpha - 1)\, y + \sin\alpha\, x)\mathbf{e}_y$.



1.2.5. Primer (d): $\mathbf{u}(x, y, z) = 10^{-4}(2\,x^2\,\mathbf{e}_x - (x + y)^2\,\mathbf{e}_y + 4\,\mathbf{e}z)$.



1.2.6. *Primer (e):* $x_1 = \sqrt{3}\, x_1^0 + x_2^0, \quad x_2 = 2\, x_2^0, \quad x_3 = x_3^0.$



1.2.7. *Komponente tenzorja velikih deformacij E*

Uporabimo enačbe [1, enačba (2.53) na str. 200] ali [2, enačba (19)]

$$
\begin{aligned}
E_{xx} &= \frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2\right),\\
E_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial y}\right),\\
E_{xz} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial z}\right),\\
E_{yy} &= \frac{\partial u_y}{\partial y} + \frac{1}{2}\left(\left(\frac{\partial u_x}{\partial y}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2\right),\\
E_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial y}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\frac{\partial u_z}{\partial z}\right),\\
E_{zz} &= \frac{\partial u_z}{\partial z} + \frac{1}{2}\left(\left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right).
\end{aligned}
$$

1.2.8. *Deformacijski gradient F*

$$
[F_{ij}] = \begin{bmatrix} \frac{\partial x_1}{\partial x_1^0} & \frac{\partial x_1}{\partial x_2^0} & \frac{\partial x_1}{\partial x_3^0} \\ \frac{\partial x_2}{\partial x_1^0} & \frac{\partial x_2}{\partial x_2^0} & \frac{\partial x_2}{\partial x_3^0} \\ \frac{\partial x_3}{\partial x_1^0} & \frac{\partial x_3}{\partial x_2^0} & \frac{\partial x_3}{\partial x_3^0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{\partial u_1}{\partial x_1^0} & \frac{\partial u_1}{\partial x_2^0} & \frac{\partial u_1}{\partial x_3^0} \\ \frac{\partial u_2}{\partial x_1^0} & \frac{\partial u_2}{\partial x_2^0} & \frac{\partial u_2}{\partial x_3^0} \\ \frac{\partial u_3}{\partial x_1^0} & \frac{\partial u_3}{\partial x_2^0} & \frac{\partial u_3}{\partial x_3^0} \end{bmatrix}
$$

$$
[F_{ij}] \overset{(a)}{=} \begin{bmatrix} 1 & a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; [F_{ij}] \overset{(b)}{=} \begin{bmatrix} 1 & -a & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; [F_{ij}] \overset{(c)}{=} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

$$
[F_{ij}] \overset{(e)}{=} \begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

1.2.9. Levi Cauchyjev tenzor $C = F^T\,F$

$$[C_{ij}] \overset{(a)}{=} \begin{bmatrix} 1+a^2 & 2\,a & 0 \\ 2\,a & 1+a^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [C_{ij}] \overset{(b)}{=} \begin{bmatrix} 1+a^2 & 0 & 0 \\ 0 & 1+a^2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$[C_{ij}] \overset{(c)}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [C_{ij}] \overset{(e)}{=} \begin{bmatrix} 3 & \sqrt{3} & 0 \\ \sqrt{3} & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1.2.10. Komponente tenzorja rotacij in vektorja rotacij v kartezijskem koordinatem sistemu (x, y, z)

Z upoštevanjem enačb [1, enačba (2.90) na str. 209] in [1, enačba (2.99) na str. 210]

$$\begin{aligned} \omega_{xy} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right), \\ \omega_{zx} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \\ \omega_{yz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \end{aligned}$$

lahko pišemo

$$[\omega_{ij}] = \begin{bmatrix} \omega_{xx} & \omega_{xy} & \omega_{xz} \\ \omega_{yx} & \omega_{yy} & \omega_{yz} \\ \omega_{zx} & \omega_{zy} & \omega_{zz} \end{bmatrix} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix},$$

$$\boldsymbol{\omega} = \omega_x \mathbf{e}_x + \omega_y \mathbf{e}_y + \omega_z \mathbf{e}_z.$$

1.2.11. Komponente tenzorja majhnih deformacij

Z upoštevanjem enačb [1, enačba (2.98) na str. 210] dobimo

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x}, \\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \\ \varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y}, \\ \varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \\ \varepsilon_{zz} &= \frac{\partial u_z}{\partial z}. \end{aligned}$$

1.2.12. Komponente tenzorja rotacij in vektorja rotacij v kartezijskem koordinatem sistemu (x, y, z)

Z upoštevanjem enačb

$$\begin{aligned} \omega_{xy} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right), \\ \omega_{zx} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \\ \omega_{yz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \end{aligned}$$

lahko pišemo

$$[\omega_{ij}] = \begin{bmatrix} \omega_{xx} & \omega_{xy} & \omega_{xz} \\ \omega_{yx} & \omega_{yy} & \omega_{yz} \\ \omega_{zx} & \omega_{zy} & \omega_{zz} \end{bmatrix} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix},$$

$$\boldsymbol{\omega} = \omega_x \mathbf{e}_x + \omega_y \mathbf{e}_y + \omega_z \mathbf{e}_z.$$

1.2.13. Komponente tenzorja velikih, majhnih deformacij in rotacij

$$[E_{ij}] = \begin{bmatrix} E_{xx} & E_{xy} & E_{xz} \\ E_{yx} & E_{yy} & E_{yz} \\ E_{zx} & E_{zy} & E_{zz} \end{bmatrix} \overset{(a)}{=} \begin{bmatrix} \frac{a^2}{2} & a & 0 \\ a & \frac{a^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [E_{ij}] \overset{(b)}{=} \begin{bmatrix} \frac{a^2}{2} & 0 & 0 \\ 0 & \frac{a^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \overset{(a)}{=} \begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\varepsilon_{ij}] \overset{(b)}{=} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$[\omega_{ij}] = \begin{bmatrix} \omega_{xx} & \omega_{xy} & \omega_{xz} \\ \omega_{yx} & \omega_{yy} & \omega_{yz} \\ \omega_{zx} & \omega_{zy} & \omega_{zz} \end{bmatrix} \overset{(a)}{=} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\omega_{ij}] \overset{(b)}{=} \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

1.2.14. Komponente tenzorja velikih, majhnih deformacij in rotacij

$$[E_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\varepsilon_{ij}] = (\cos\alpha - 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[\omega_{ij}] = \sin\alpha \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\boldsymbol{\omega} = \sin\alpha \mathbf{e}_z.$$

1.2.15. Specifična sprememba dolžine

Obravnavali bomo samo primer (*a*). Izhajamo iz enačb [1, enačba (2.64) na str. 202] ali [2, enačba (23)]. $D_{\alpha\alpha} = \sqrt{1 + 2\,E_{\alpha\alpha}} - 1 \approx E_{\alpha\alpha} \approx \varepsilon_{\alpha\alpha}$.

$$D_{AB} = \frac{|AB'| - |AB|}{|AB|} = \frac{\sqrt{1 + a^2} - 1}{1} = D_{xx} \approx E_{xx} = \frac{a^2}{2} \approx \varepsilon_{xx} = 0,$$

$$D_{AC} = \frac{|AC'| - |AC|}{|AC|} = \frac{\sqrt{1 + a^2} - 1}{1} = D_{yy} \approx E_{yy} = \frac{a^2}{2} \approx \varepsilon_{yy} = 0.$$

Z uvedbo enotskih vektorjev $\boldsymbol{e}_\xi = \frac{\sqrt{2}}{2}(\boldsymbol{e}_x + \boldsymbol{e}_y)$ in $\boldsymbol{e}_\eta = \frac{\sqrt{2}}{2}(-\boldsymbol{e}_x + \boldsymbol{e}_y)$ v smereh AC in BD, komponent $E_{\xi\xi} = E_{xx}\,e_{\xi x}^2 + E_{yy}\,e_{\xi y}^2 + 2\,E_{xy}\,e_{\xi x}\,e_{\xi y} = a + \frac{a^2}{2}$ in $E_{\eta\eta} = E_{xx}\,e_{\eta x}^2 + E_{yy}\,e_{\eta y}^2 + 2\,E_{xy}\,e_{\eta x}\,e_{\eta y} = -a + \frac{a^2}{2}$ dobimo

$$D_{AD} = \frac{|AD'| - |AD|}{|AD|} = \frac{\sqrt{2}(1 + a) - \sqrt{2}}{\sqrt{2}} = a = D_{\xi\xi} \approx E_{\xi\xi} \approx \varepsilon_{\xi\xi} = a,$$

$$D_{BC} = \frac{|B'C'| - |BC|}{|BC|} = \frac{\sqrt{2}(1 - a) - \sqrt{2}}{\sqrt{2}} = -a = D_{\eta\eta} \approx E_{\eta\eta} \approx \varepsilon_{\eta\eta} = -a.$$

1.2.16. Polarni razcep deformacijskega gradienta v primeru (e)

$$[U_{ij}] = \frac{1}{2\sqrt{2}} \begin{bmatrix} 3 + \sqrt{3} & 3 - \sqrt{3} & 0 \\ 3 - \sqrt{3} & 1 + 3\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix}$$

$$[U_{ij}]^{-1} = \frac{1}{4\sqrt{6}} \begin{bmatrix} 1 + 3\sqrt{3} & \sqrt{3} - 3 & 0 \\ \sqrt{3} - 3 & 3 + \sqrt{3} & 0 \\ 0 & 0 & 4\sqrt{6} \end{bmatrix}$$

$$[R_{ij}] = \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3} + 1 & \sqrt{3} - 1 & 0 \\ 1 - \sqrt{3} & \sqrt{3} + 1 & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix}$$

$$[V_{ij}] = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{3} + 1 & \sqrt{3} - 1 & 0 \\ \sqrt{3} - 1 & \sqrt{3} + 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

1.2.17. Sprememba pravega kota

Ponovno bomo obravnavali samo primer (a). Izhajamo iz enačb [1, enačba (2.70) na str. 204] ali [2, enačba (25)]. $D_{\alpha\beta} = \arcsin\left(\frac{2\,E_{\alpha\beta}}{\sqrt{1+2\,E_{\alpha\alpha}}\,\sqrt{1+2\,E_{\beta\beta}}}\right) \approx 2\,E_{\alpha\beta} \approx 2\,\varepsilon_{\alpha\beta}$. Označimo spremembi pravih kotov z D_{CAB} in z D_{CED}. Z upoštevanjem slike dobimo

$$\begin{aligned}
\sin(D_{CAB}) &= \sin(2\,\alpha) = 2\,\sin\alpha\,\cos\alpha \\
&= \frac{2\,E_{xy}}{\sqrt{1+2\,E_{xx}}\,\sqrt{1+2\,E_{yy}}} = \frac{2\,a}{\sqrt{1+a^2}\,\sqrt{1+a^2}}, \\
&= D_{xy} \approx 2\,E_{xy} \approx 2\,\varepsilon_{xy} = 2\,a.
\end{aligned}$$

Spremembo pravega kota CAB zapišemo z $D_{\xi\eta} = \arcsin\left(\frac{2\,E_{\xi\eta}}{\sqrt{1+2\,E_{\xi\xi}}\,\sqrt{1+2\,E_{\eta\eta}}}\right)$. Izračunamo $E_{\xi\eta} = E_{xx}\,e_{\xi x}\,e_{\eta x} + E_{xy}\,e_{\xi x}\,e_{\eta y} + E_{yx}\,e_{\xi y}\,e_{\eta x} + E_{yy}\,e_{\xi y}\,e_{\eta y} = 0$ in $\varepsilon_{\xi\eta} = \varepsilon_{xx}\,e_{\xi x}\,e_{\eta x} + \varepsilon_{xy}\,e_{\xi x}\,e_{\eta y} + \varepsilon_{yx}\,e_{\xi y}\,e_{\eta x} + \varepsilon_{yy}\,e_{\xi y}\,e_{\eta y} = 0$. Posledično je $D_{\xi\eta} = 0 = 2\,E_{\xi\eta} = 2\,\varepsilon_{\xi\eta}$.

1.2.18. Glavne normalne deformacije v primeru (a)

Glavne normalne deformacije so kar lastne vrednosti matrike $[\varepsilon_{ij}]$. To so ničle polinoma

$$\begin{vmatrix} -\lambda & a & 0 \\ a & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda\,(\lambda^2 - a^2) = -\lambda\,(\lambda - a)\,(\lambda + a) = 0.$$

$$\begin{aligned}
\lambda_1 &= a = \varepsilon_{11}, \\
\lambda_2 &= 0 = \varepsilon_{22}, \\
\lambda_3 &= -a = \varepsilon_{33}.
\end{aligned}$$

Smeri glavnih ravnin so določene s pripadajočimi lastnimi vektorji matrike $[\varepsilon_{ij}]$.

Lastni vektor, ki pripada lastni vrednosti $\varepsilon_{11} = a$ reši enačbo

$$\begin{aligned}
([\varepsilon_{ij}] - \varepsilon_{11}\,[I])\,\boldsymbol{x} &= \mathbf{0} \\
([\varepsilon_{ij}] - a\,[I])\,\boldsymbol{x} &= \mathbf{0} \\
\begin{bmatrix} -a & a & 0 \\ a & -a & 0 \\ 0 & 0 & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Splošna rešitev gornje enačbe se glasi

$$\boldsymbol{x} = \begin{bmatrix} \alpha & \alpha & 0 \end{bmatrix}^T, \quad 0 \neq \alpha \in R.$$

Izberemo en sam bazni vektor

$$\mathbf{e}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}^T,$$

ki se ujema z vektorjem $\mathbf{e}_\xi$.

Lastni vektor, ki pripada lastni vrednosti $\varepsilon_{22} = 0$ reši enačbo

$$\begin{aligned}
([\varepsilon_{ij}] - \varepsilon_{22}\,[I])\,\boldsymbol{x} &= \mathbf{0} \\
([\varepsilon_{ij}] - 0\,[I])\,\boldsymbol{x} &= \mathbf{0} \\
\begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Splošna rešitev gornje enačbe se glasi

$$\boldsymbol{x} = \begin{bmatrix} 0 & 0 & \alpha \end{bmatrix}^T, \quad 0 \neq \alpha \in R.$$

Izberemo en sam bazni vektor

$$\mathbf{e}_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T,$$

ki se ujema z vektorjem $\mathbf{e}_z$.

Lastni vektor, ki pripada lastni vrednosti $\varepsilon_{33} = -a$ reši enačbo

$$\begin{gathered}
([\varepsilon_{ij}] - \varepsilon_{33}\,[I])\,\boldsymbol{x} = \mathbf{0} \\
([\varepsilon_{ij}] + a\,[I])\,\boldsymbol{x} = \mathbf{0} \\
\begin{bmatrix} a & a & 0 \\ a & a & 0 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{gathered}$$

Splošna rešitev gornje enačbe se glasi

$$\boldsymbol{x} = \begin{bmatrix} -\alpha & \alpha & 0 \end{bmatrix}^T, \quad 0 \neq \alpha \in R.$$

Izberemo en sam bazni vektor

$$\mathbf{e}_3 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}^T,$$

ki se ujema z vektorjem $\mathbf{e}_\eta$.

1.2.19. Rešitev v Matlabu

Literatura

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