10. Vaja: Hitrosti napetosti

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1. O objektivnosti tenzorjev

1.1. Rotacija

1.1.1. Superpozicija transformacij

$$F = F_2\, F_1. \tag{1}$$

$$\begin{aligned} R^+\, U^+ &= F^+ = Q\, F = Q\, R\, U, \\ R^+ &= Q\, R, \\ U^+ &= U. \end{aligned} \tag{2}$$

$$\begin{aligned} V^+\, R^+ &= F^+ = Q\, F = Q\, V\, R, \\ R^+ &= Q\, R, \\ V^+ &= Q\, V\, Q^T. \end{aligned} \tag{3}$$

1.2. Dva opazovalca

2. Objektivni odvodi napetosti po času

2.1. Korotacijski ali Jaumannov odvod

Spin

$$\begin{aligned} W^+ &= \Omega + Q\, W\, Q^T \\ &= \dot{Q}\, Q^T + Q\, W\, Q^T. \end{aligned}$$

2.1.1. Odvod vektorja

Korotacijski odvod

$$\begin{aligned} \dot{u^+} - W^+\, u^+ &= (\dot{u^+}) - W^+\, u^+ \\ &= \dot{Q}\, u + Q\, \dot{u} - \dot{Q}\, u - Q\, W\, u \\ &= (\dot{u} - W\, u)^+ = (\overset{\circ}{u})^+ \\ &= Q\, (\dot{u} - W\, u) = Q\ \overset{\circ}{u}\ . \end{aligned}$$

Korotacijski odvod

$$\begin{aligned} \overset{\circ}{u} &= \dot{u} - Q\, u, \\ (\overset{\circ}{u})^+ &= Q\ \overset{\circ}{u}\ . \end{aligned}$$

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2.1.2. Odvod tenzorja 2 reda

Iz enačb

$$A^+ = Q\,A\,Q^T,$$
$$\dot{A^+} = \dot{Q}\,A\,Q^T + Q\,\dot{A}\,Q^T + Q\,A\,\dot{Q}^T,$$
$$W^+ = \Omega + Q\,W\,Q^T = \dot{Q}\,Q^T + Q\,W\,Q^T \to W^T\,Q = \dot{Q} + Q\,W \to \dot{Q} = W^T\,Q - Q\,W$$

dobimo zvezo

Jaumannov odvod

$$(\dot{A} - W\,A + A\,W)^+ = Q\,(\dot{A} - W\,A + A\,W)\,Q^T,$$
$$\overset{\circ}{A} = \dot{A} - W\,A + A\,W,$$
$$(\overset{\circ}{A})^+ = Q\ \overset{\circ}{A}\ Q^T.$$

Lastnosti Jaumannovega odvoda

$$2\,A : \overset{\circ}{A} = 2\,A : \dot{A} - 2\,A : (W\,A) + 2\,A : (A\,W) = 2\,A : \dot{A}.$$

2.2. Konvekcijski odvod

$$L^+ = \Omega + Q\,L\,Q^T.$$

Konvekcijski odvod

$$\overset{\triangle}{u} = \dot{u} + L^T\,u.$$

2.2.1. Odvod vektorja

2.2.2. Odvod tenzorja 2 reda

$$\overset{\triangle}{A} = \dot{A} + L^T\,A + A\,L$$
$$(\overset{\triangle}{A})^+ = Q\ \overset{\triangle}{A}\ Q^T.$$

3. Objektivni časovni odvodi tenzorjev napetosti

3.1. Jaumannov odvod

$$\overset{\nabla}{\sigma} = \dot{\sigma} - W\,\sigma + \sigma\,W.$$

3.2. Truesdellov odvod

$$\overset{\circ}{\sigma} = \dot{\sigma} - L\,\sigma - \sigma\,L^T + \mathrm{tr}(L)\,\sigma.$$

3.3. Green–Naghdijev odvod

$$\overset{\triangle}{\sigma} = \dot{\sigma} - \dot{R}\,R^T\,\sigma + \sigma\,\dot{R}\,R^T.$$

3.4. Oldroydov odvod

$$\overset{\circ}{\sigma} = \dot{\sigma} - L\,\sigma - \sigma\,L^T.$$

4. Čisti strig

Obravnavaj spodnje deformiranje telesa:

$$\begin{aligned} x_1 &= x_1^0 + t\, x_2^0, \\ x_2 &= x_2^0. \end{aligned} \tag{4}$$

4.1. Deformacijski gradient F in inverz deformacijskega gradienta F^{-1}

$$F = \begin{bmatrix} \dfrac{\partial x_1}{\partial x_1^0} & \dfrac{\partial x_1}{\partial x_2^0} \\ \dfrac{\partial x_2}{\partial x_1^0} & \dfrac{\partial x_2}{\partial x_2^0} \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \Longrightarrow F^{-1} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}. \tag{5}$$

4.2. Hitrost $\vec{v}$

$$v_1 = \frac{D\, x_1}{D\, t} = x_2^0, \quad v_2 = \frac{D\, x_2}{D\, t} = 0. \tag{6}$$

4.3. $\dot{F}$, L, D in W

$$\dot{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{7}$$

$$\dot{F} = L\, F \Longrightarrow L = \dot{F}\, F^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{8}$$

$$D = \frac{1}{2}\,(L + L^T) = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{9}$$

$$W = \frac{1}{2}\,(L - L^T) = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{10}$$

4.4. Jaumannov odvod

$$\overset{\nabla}{\sigma} = \dot{\sigma} - W\,\sigma + \sigma\, W = C : D = \lambda\, \mathrm{tr}(D)\, I + 2\,\mu\, D \Longrightarrow$$

$$\dot{\sigma} = \lambda\, \mathrm{tr}(D)\, I + 2\,\mu\, D + W\,\sigma - \sigma\, W$$

$$\begin{aligned} \begin{bmatrix} \dot{\sigma}_{xx} & \dot{\sigma}_{xy} & \dot{\sigma}_{xz} \\ \dot{\sigma}_{yx} & \dot{\sigma}_{yy} & \dot{\sigma}_{yz} \\ \dot{\sigma}_{zx} & \dot{\sigma}_{zy} & \dot{\sigma}_{zz} \end{bmatrix} &= \lambda\, \mathrm{tr}\left(\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) I + 2\,\mu \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &+ \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \mu & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \sigma_{xy} & -\sigma_{xx} & 0 \\ \sigma_{yy} & -\sigma_{yx} & 0 \\ \sigma_{zy} & -\sigma_{zx} & 0 \end{bmatrix} \end{aligned}$$

4.4.1. Sistem linearnih diferencialnih enačb

Od tu izpeljemo z upoštevanjem simetrije tenzorja σ sistem linearnih diferencialnih enačb:

$$\dot{\sigma}_{xx} = \sigma_{xy} \tag{11a}$$

$$\dot{\sigma}_{xy} = \mu + \frac{1}{2}(\sigma_{yy} - \sigma_{xx}) \tag{11b}$$

$$\dot{\sigma}_{yy} = -\sigma_{xy} \tag{11c}$$

$$\dot{\sigma}_{xz} = \frac{1}{2}\sigma_{yz} \tag{11d}$$

$$\dot{\sigma}_{yz} = -\frac{1}{2}\sigma_{xz} \tag{11e}$$

$$\dot{\sigma}_{zz} = 0 \tag{11f}$$

4.4.2. Reševanje sistema linearnih diferencialnih enačb

Odvajamo enačbo (11e), upoštevamo enačbo (11d)

4.4.3. Rešitev sistema linearnih diferencialnih enačb