Naloga

Pomik tanke stene stranico $b = 1\,\text{m}$ debeline $d = 1\,\text{cm}$ je podan z vektorji

$$
\begin{aligned}
(a)\quad & u(x,y,z) = u_x e_x + u_y e_y = a y e_x + a x e_y \Longrightarrow u_x = a y,\ u_y = a x, \\
(b)\quad & u(x,y,z) = u_x e_x + u_y e_y = -a y e_x + a x e_y \Longrightarrow u_x = -a y,\ u_y = a x \\
(c)\quad & u(x,y,z) = \text{ rotacija okrog osi } z \text{ za kot } \alpha.
\end{aligned}
$$

Določi:

- Komponente tenzorja majhnih deformacij ε_{ij}, komponente tenzorja velikih deformacij E_{ij}, komponente tenzorja rotacij ω_{ij} v kartezičnem koordinatnem sistemu in vektor zasuka $\boldsymbol{\omega}$.
- Specifične spremembe dolžin daljic AB, AC, AD in BC in spremembi pravih kotov CAB in CED.
- Glavne normalne deformacije in pripadajoče smeri.

Podatki: $a = 10^{-4}$, $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$, $D(1,1,0)$ in $E(0.5,0.5,0)$. Vse razdalje so v metrih.

Pomik v primeru (c)

Z upoštevanjem enačbe

$$u = \sin\omega\, e_\omega \times r + (1 - \cos\omega)\, e_\omega \times (e_\omega \times r)$$

in podatkov

$$\omega = \alpha, \quad e_\omega = e_z, \quad r = xe_x + ye_y,$$
$$e_z \times r = xe_y - ye_x, \quad e_z \times (e_z \times r) = -xe_x - ye_y,$$

lahko pomik zapišemo z enačbo

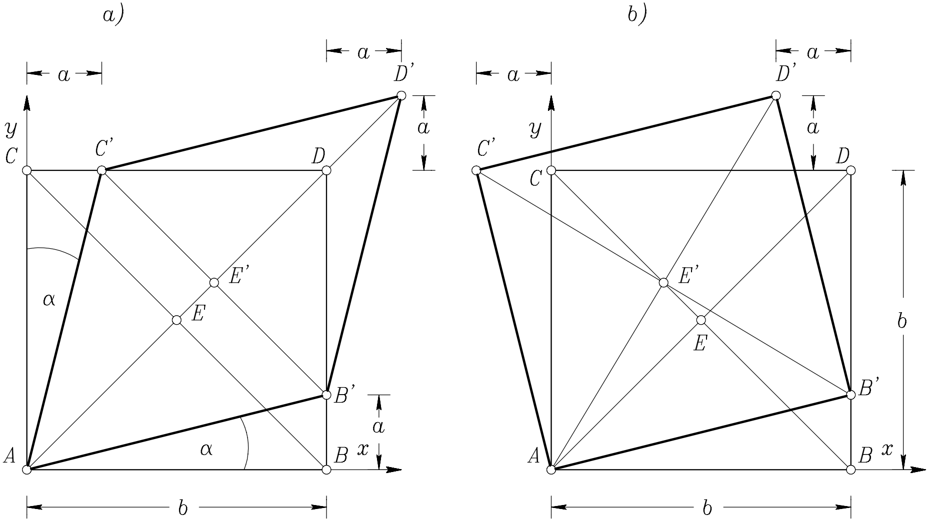
$$\begin{aligned} u &= \sin\alpha\, e_z \times r + (1 - \cos\alpha)\, e_z \times (e_z \times r), \\ &= (-x(1 - \cos\alpha) - y\sin\alpha)e_x + (x\sin\alpha - y(1 - \cos\alpha))e_y \end{aligned}$$

oziroma po komponentah

$$u_x = -x(1 - \cos\alpha) - y\sin\alpha, \quad u_y = x\sin\alpha - y(1 - \cos\alpha).$$

Deformacija stene v primerih (a) in (b)

Prvotno nedeformirano stanje stene na sliki podajata kvadrata z oglišči A, B, C in D, deformirano pa paralelograma oglišči A, B', C' in D'.



Komponente tenzorja velikih deformacij v kartezijskem koordinatem sistemu (x, y, z)

$$E_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2\right),$$

$$E_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial y}\right),$$

$$E_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial z}\right),$$

$$E_{yy} = \frac{\partial u_y}{\partial y} + \frac{1}{2}\left(\left(\frac{\partial u_x}{\partial y}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2\right),$$

$$E_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial y}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\frac{\partial u_z}{\partial z}\right),$$

$$E_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2}\left(\left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right).$$

Komponente tenzorja majhnih deformacij v kartezijskem koordinatem sistemu (x, y, z)

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_x}{\partial x}, \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \\
\varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \\
\varepsilon_{yy} &= \frac{\partial u_y}{\partial y}, \\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z}.
\end{aligned}
$$

Komponente tenzorja rotacij in vektorja rotacij v kartezijskem koordinatem sistemu (x, y, z)

Z upoštevanjem enačb

$$\begin{aligned}
\omega_{xy} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right), \\
\omega_{zx} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \\
\omega_{yz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)
\end{aligned}$$

lahko pišemo

$$[\omega_{ij}] = \begin{bmatrix} \omega_{xx} & \omega_{xy} & \omega_{xz} \\ \omega_{yx} & \omega_{yy} & \omega_{yz} \\ \omega_{zx} & \omega_{zy} & \omega_{zz} \end{bmatrix} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix},$$

$$\boldsymbol{\omega} = \omega_x e_x + \omega_y e_y + \omega_z e_z.$$

Komponente tenzorjev velikih, majhnih deformacij in rotacij v primerih (a) in (b) v kartezijskem koordinatem sistemu (x,y,z)

$$[E_{ij}] = \begin{bmatrix} E_{xx} & E_{xy} & E_{xz} \\ E_{yx} & E_{yy} & E_{yz} \\ E_{zx} & E_{zy} & E_{zz} \end{bmatrix} \overset{(a)}{=} \begin{bmatrix} \frac{a^2}{2} & a & 0 \\ a & \frac{a^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [E_{ij}] \overset{(b)}{=} \begin{bmatrix} \frac{a^2}{2} & 0 & 0 \\ 0 & \frac{a^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \overset{(a)}{=} \begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\varepsilon_{ij}] \overset{(b)}{=} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$[\omega_{ij}] = \begin{bmatrix} \omega_{xx} & \omega_{xy} & \omega_{xz} \\ \omega_{yx} & \omega_{yy} & \omega_{yz} \\ \omega_{zx} & \omega_{zy} & \omega_{zz} \end{bmatrix} \overset{(a)}{=} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\omega_{ij}] \overset{(b)}{=} \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Komponente tenzorjev velikih, majhnih deformacij in rotacij v primeru (c) v kartezijskem koordinatem sistemu (x, y, z)

$$[E_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\varepsilon_{ij}] = (\cos\alpha - 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[\omega_{ij}] = \sin\alpha \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\boldsymbol{\omega} = \sin\alpha \boldsymbol{e}_z.$$

Specifična sprememba dolžin stranic AB, AC, AD in BC

Obravnavali bomo samo primer (a). Izhajamo iz enačb
$D_{\alpha\alpha} = \sqrt{1+2E_{\alpha\alpha}} - 1 \approx E_{\alpha\alpha} \approx \varepsilon_{\alpha\alpha}$.

$$D_{AB} = \frac{|AB'|-|AB|}{|AB|} = \frac{\sqrt{1+a^2}-1}{1} = D_{xx} \approx E_{xx} = \frac{a^2}{2} \approx \varepsilon_{xx} = 0,$$

$$D_{AC} = \frac{|AC'|-|AC|}{|AC|} = \frac{\sqrt{1+a^2}-1}{1} = D_{yy} \approx E_{yy} = \frac{a^2}{2} \approx \varepsilon_{yy} = 0.$$

Z uvedbo enotskih vektorjev $e_\xi = \frac{\sqrt{2}}{2}(e_x+e_y)$ in $e_\eta = \frac{\sqrt{2}}{2}(-e_x+e_y)$ v smereh AC in BD, komponent
$E_{\xi\xi} = E_{xx}\,e_{\xi x}^2 + E_{yy}\,e_{\xi y}^2 + 2E_{xy}\,e_{\xi x}\,e_{\xi y} = a + \frac{a^2}{2}$ in
$E_{\eta\eta} = E_{xx}\,e_{\eta x}^2 + E_{yy}\,e_{\eta y}^2 + 2E_{xy}\,e_{\eta x}\,e_{\eta y} = -a + \frac{a^2}{2}$ dobimo

$$D_{AD} = \frac{|AD'|-|AD|}{|AD|} = \frac{\sqrt{2}(1+a)-\sqrt{2}}{\sqrt{2}} = a = D_{\xi\xi} \approx E_{\xi\xi} \approx \varepsilon_{\xi\xi} = a,$$

$$D_{BC} = \frac{|B'C'|-|BC|}{|BC|} = \frac{\sqrt{2}(1-a)-\sqrt{2}}{\sqrt{2}} = -a = D_{\eta\eta} \approx E_{\eta\eta} \approx \varepsilon_{\eta\eta} = -a.$$

Spremembi pravih kotov *CAB* in *CED*

Ponovno bomo obravnavali samo primer (a). Izhajamo iz enačb $D_{\alpha\beta} = \arcsin\left(\frac{2E_{\alpha\beta}}{\sqrt{1+2E_{\alpha\alpha}}\sqrt{1+2E_{\beta\beta}}}\right) \approx 2E_{\alpha\beta} \approx 2\varepsilon_{\alpha\beta}$. Označimo spremembi pravih kotov z D_{CAB} in z D_{CED}. Z upoštevanjem slike dobimo

$$\begin{aligned}
\sin(D_{CAB}) &= \sin(2\alpha) = 2\sin\alpha\cos\alpha \\
&= \frac{2E_{xy}}{\sqrt{1+2E_{xx}}\sqrt{1+2E_{yy}}} = \frac{2a}{\sqrt{1+a^2}\sqrt{1+a^2}}, \\
&= D_{xy} \approx 2E_{xy} \approx 2\varepsilon_{xy} = 2a.
\end{aligned}$$

Spremembo pravega kota *CAB* zapišemo z $D_{\xi\eta} = \arcsin\left(\frac{2E_{\xi\eta}}{\sqrt{1+2E_{\xi\xi}}\sqrt{1+2E_{\eta\eta}}}\right)$. Izračunamo $E_{\xi\eta} = E_{xx}e_{\xi x}e_{\eta x} + E_{xy}e_{\xi x}e_{\eta y} + E_{yx}e_{\xi y}e_{\eta x} + E_{yy}e_{\xi y}e_{\eta y} = 0$ in $\varepsilon_{\xi\eta} = \varepsilon_{xx}e_{\xi x}e_{\eta x} + \varepsilon_{xy}e_{\xi x}e_{\eta y} + \varepsilon_{yx}e_{\xi y}e_{\eta x} + \varepsilon_{yy}e_{\xi y}e_{\eta y} = 0$. Posledično je $D_{\xi\eta} = 0 = 2E_{\xi\eta} = 2\varepsilon_{\xi\eta}$.

Glavne normalne deformacije v primeru (a)

Glavne normalne deformacije so kar lastne vrednosti matrike $[\varepsilon_{ij}]$. To so ničle polinoma

$$\begin{vmatrix} -\lambda & a & 0 \\ a & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda\left(\lambda^2 - a^2\right) = -\lambda\left(\lambda - a\right)\left(\lambda + a\right) = 0.$$

$$\begin{aligned} \lambda_1 &= a = \varepsilon_{11}, \\ \lambda_2 &= 0 = \varepsilon_{22}, \\ \lambda_3 &= -a = \varepsilon_{33}. \end{aligned}$$

Smeri glavnih ravnin

Smeri glavnih ravnin so določene s pripadajočimi lastnimi vektorji matrike $[\varepsilon_{ij}]$.
Lastni vektor, ki pripada lastni vrednosti $\varepsilon_{11} = a$ reši enačbo

$$\left([\varepsilon_{ij}] - \varepsilon_{11}\,[I]\right)x = 0$$

$$\left([\varepsilon_{ij}] - a\,[I]\right)x = 0$$

$$\begin{bmatrix} -a & a & 0 \\ a & -a & 0 \\ 0 & 0 & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Splošna rešitev gornje enačbe se glasi

$$x = \begin{bmatrix} \alpha & \alpha & 0 \end{bmatrix}^T, \quad 0 \neq \alpha \in R.$$

Izberemo en sam bazni vektor

$$e_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}^T,$$

ki se ujema z vektorjem e_ξ.

Smeri glavnih ravnin

Lastni vektor, ki pripada lastni vrednosti $\varepsilon_{22} = 0$ reši enačbo

$$\left([\varepsilon_{ij}] - \varepsilon_{22}\,[I]\right)x = 0$$
$$\left([\varepsilon_{ij}] - 0\,[I]\right)x = 0$$
$$\begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Splošna rešitev gornje enačbe se glasi

$$x = \begin{bmatrix} 0 & 0 & \alpha \end{bmatrix}^T, \quad 0 \neq \alpha \in R.$$

Izberemo en sam bazni vektor

$$e_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T,$$

ki se ujema z vektorjem e_z.

Smeri glavnih ravnin

Lastni vektor, ki pripada lastni vrednosti $\varepsilon_{33} = -a$ reši enačbo

$$\left([\varepsilon_{ij}] - \varepsilon_{33}\,[I]\right) x = 0$$

$$\left([\varepsilon_{ij}] + a\,[I]\right) x = 0$$

$$\begin{bmatrix} a & a & 0 \\ a & a & 0 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Splošna rešitev gornje enačbe se glasi

$$x = \begin{bmatrix} -\alpha & \alpha & 0 \end{bmatrix}^T, \quad 0 \neq \alpha \in R.$$

Izberemo en sam bazni vektor

$$e_3 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}^T,$$

ki se ujema z vektorjem e_η.